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RECOGNITION AND OPTIMIZATION ALGORITHMS IN $\{P_6,C_6\}$-FREE GRAPHS

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Secure dominating sets can be applied as protection strategies by minimizing the number of guards to secure a system so as to be cost effective as possible.
Dominating sets that induce a complete subgraph have a great diversity of applications. In setting up the communications links in a network one might want a strong core group that can communicate with each other member of the core group and so that everyone from outside the group could communicate with someone within the core group.
For arbitrary graphs, the problem of finding the size of a minimum dominating set in the graph is an NP-complete problem. The dominating set problem remains NP-complete even for some specific classes of graphs, including chordal graphs, split graphs and bipartite graphs.
The class of P_k-free graphs has received a fair amount of attention in the theory of graph algorithms. Given an NP-hard optimization problem, it is often fruitful to study its complexity when the instances are restricted to P_k-free graphs.
An algorithm that finds such a dominating subgraph of a connected P_6-free graph in polynomial time enables us to solve the hypergraph 2-colorability problem in polynomial time for the class of hypergraphs with P_6-free incidence graphs.
We characterize the triangle-free and $\{P_6, C_6\}$ -free graphs using weakly decomposition and a dominating complete bipartite subgraph, we give a recognition algorithm, and combinatorial optimization algorithms for this class of graphs, comparable from the point of view of execution in time with existing ones.

Keywords: $\{P_6,C_6\}$-free graphs, weak decomposition, recognition algorithms, combinatorial optimization algorithms.

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INTRODUCTION

We recall some results concerning the P_6-free graphs.

In ([2]) the authors give a complete structure description of (prime) (P_6,K_3)-free graphs.

In ([7]) the author prove that on the class of $(P_6,\text{diamond})$-free graphs the Maximum-Weight

Independent Set problem and the Minimum-Weight Independent Dominating Set problem can be solved in polynomial time. In ([11]) it is stated that the following problems are executed in polynomial time: recognition, domination are polynomial.

The content of the paper is organized as follows. In Preliminaries, we give the usual terminology in graph theory. In Section 3, we give the results.

PRELIMINARIES

Throughout this paper, $G=(V,E)$ is a connected, finite and undirected graph ([1]), without loops and multiple edges, having $V=V(G)$ as the vertex set and $E=E(G)$ as the set of edges. $\overline{G}$ (*co-G*) is the complement of G. If $U\subseteq V$, by $G(U)$ we denote the subgraph of G induced by U. By $G-X$ we mean the subgraph $G(V-X)$, whenever $X\subseteq V$, but we simply write $G-v$, when $X=\{v\}$. If $e=xy$ is an edge of a graph G, then x and y are adjacent, while x and e are incident, as are y and e. If $xy\in E$, we also use $x\sim y$, and $x\nsim y$ whenever x, y are not adjacent in G. If A, $B\subseteq V$ are disjoint and $ab\in E$ for every $a\in A$ and $b\in B$, we say that A, B are *totally adjacent* and we denote by $A\sim B$, while by $A\nsim B$ we mean that no edge of G joins some vertex of A to a vertex from B and, in this case, we say A and B are *totally non-adjacent*.

The *neighborhood* of the vertex $v\in V$ is the set $N_G(v)=\{u\in V:uv\in E\}$, while $N_G[v]=\{v\}\cup N_G(v)$; we denote $N(v)$ and $N[v]$, when G appears clearly from the context. The *degree* of v in G is $d_G(v)=|N_G(v)|$. The neighborhood of the vertex v in the complement of G will be denoted by $\overline{N}(v)$.

The neighborhood of $S\subseteq V$ is the set $N(S)=\cup_{v\in S}N(v)-S$ and $N[S]=S\cup N(S)$. A graph is complete if every pair of distinct vertices is adjacent.

The *chromatic number* of a graph G ($\chi(G)$) is the minimum number of colors needed to label all its vertices in such a way that no two vertices with the same color are adjacent. The *stability number* $\alpha(G)$ of a graph G is the size of the largest stable set. An independent (stable) set of a graph G is a subset of pairwise non-adjacent vertices. The *clique number* of a graph G is a number of the vertices in a maximum clique of G, denoted by $\omega(G)$. A *dominating set* of a graph G is a subset D of its vertices, such that every vertex not in D is adjacent to at least one member of D. A *minimum dominating set* of G has the minimum cardinality among all dominating sets of G. The domination number $\gamma(G)$ of G is the cardinality of a minimum dominating set of G. A *clique cover* of a graph $G=(V,E)$ is a partition P of V such that each part in P induces a clique in G. The *minimum clique cover* of G, $\theta(G)$, is the minimum number of parts in a clique cover of G. Note that the clique cover number of a graph is exactly the chromatic number of its complement.

By P_n, C_n, K_n we mean a chordless path on $n\geq 3$ vertices, a chordless cycle on $n\geq 3$ vertices, and a complete graph on $n\geq 1$ vertices, respectively.

Let F denote a family of graphs. A graph G is called F *-free* if none of its subgraphs are in F .

The *sum* of the graphs G_1, G_2 is the graph $G=G_1+G_2$ having:

$V(G)=V(G_1)\cup V(G_2)$, $E(G)=E(G_1)\cup E(G_2)\cup \{uv:u\in V(G_1),v\in V(G_2)\}$.

If G_1 and G_2 are two graphs with $V(G_1)\cap V(G_2)=\phi$ then we set $G=G_1\cup G_2$ with $V(G)=V(G_1)\cup V(G_2)$ and $E(G)=E(G_1)\cup E(G_2)$.

THE RESULTS

1.0. Known results

We remind a characterization of the weak decomposition of a graph.

Definition 1. ([9], [10]) *A set $A\subseteq V(G)$ is called a weak set of the graph G if $N_G(A)\neq V(G)-A$ and $G(A)$ is connected. If A is a weak set, maximal with respect to set inclusion, then $G(A)$ is called a weak component. For simplicity, the weak component $G(A)$ will be denoted with A.*

Definition 2. ([9], [10]) *Let $G=(V,E)$ be a connected and non-complete graph. If A is a weak set, then the partition $\{A,N(A),V-A\cup N(A)\}$ is called a weak decomposition of G with respect to A.*

The name of *weak component* is justified by the following result.

Theorem 1. ([9], [10]) *Every connected and non-complete graph $G=(V,E)$ admits a weak component A such that $G(V-A)=G(N(A))+G(\overline{N}(A))$.*

Theorem 2. ([3], [4]) *Let $G=(V,E)$ be a connected and non-complete graph and $A\subseteq V$. Then A is a weak component of G if and only if $G(A)$ is connected and $N(A)\sim\overline{N}(A)$.*

The next result, that follows from Theorem 2, ensures the existence of a weak decomposition in a connected and non-complete graph.

Corollary 1. *If $G=(V,E)$ is a connected and non-complete graph, then V admits a weak decomposition*

(A,B,C), such that G(A) is a weak component and G(V-A)=G(B)+G(C).

Theorem 2 provides an *O(n+m)* algorithm for building a weak decomposition for a non-complete and connected graph.

Algorithm for the weak decomposition of a graph ([9])

Input: A connected graph with at least two nonadjacent vertices, G=(V,E).

Output: A partition V=(A,N,R) such that G(A) is connected, N=N(A), A $\not\sim$R=$\overline{N}$ (A).

Begin

A:= any set of vertices such that A∪N(A)≠ V N:= N(A)

R := V-A∪N(A)

While (∃n∈N, ∃r∈R such that nr∉E) *do*

Begin

A:= A∪{n}

N:= (N-{n})∪ (N(n)∩ R)

R:=R-(N(n) ∩R)

end

end

1.1. Characterization of $\{P_6, C_6\} - free$ and triangle-free graphs

The following result is known.

Theorem 1. ([5]) *Let G be a triangle-free graph. Then G is both P_6-free and C_6-free if and only if every connected induced subgraph of G has a dominating complete bipartite subgraph.*

Next to characterize the $\{P_6, C_6\} - free$ graphs we using weak decomposition.

For this we showed the next lemma.

Lemma 3.1.1. *If G is $\{P_6, C_6, C_3\} - free$ graph and (A,N,R) is a weak decomposition with G(A) weak component then N and R are stable set and $N_G(N) - R = A$.*

Proof. Since *G(A)* is the weak component it follows that $N \sim R$. If *G(R)* would contain an xy edge then $G(\{n, x, y\}) \cong C_3, \forall n \in N,$ contrary to the fact that G is $C_3 - free.$ So R is a stable set. Similarly, N is a stable set. Since $N \sim R$ it follows that $G(N \cup R)$ it is a complete bipartite graph.

We show: $\forall a \in A, \exists n_0 \in N$ such that $an_0 \in E.$

We assume that $\exists a_0 \in A$ such that $\forall n \in N: a_0 n \notin E$ (i.e. $\{a_0\} \not\sim N$). Is known that N=N_G(A). So, $\forall n' \in N, \exists a' \in A$ such that $a'n' \in E.$ Since G(A) is connected, it follow that $\exists L_{a'a_0}.$ Going the path $L_{a'a_0}$ in the sense of a' to a_0, there is a last vertex a_1' such that $a_1'n' \in E.$ Let a_0" be the first vertex on $L_{a'a_0}$, in the sense of going from a' to a_0, such that $\{a_0''\} \not\sim N$ ($\exists a_0''$, possible $a_0'' = a_0$).

Given that the connected induced subgraph, $H_0 = G(V(L_{a_0''a_1'}) \cup N \cup R),$ let $\forall H_1$ be induced complete bipartite subgraph of H_0. We have $H_1 = G(N_1 \cup R_1)$ where $N_1 \subseteq N, R_1 \subseteq R.$ (Indeed, if $V(H_1) \supseteq V(L_{a_0''a_1'})$ then H_1 is not a complete bipartite subgraph).

But $V(H_1)(= N_1 \cup R_1)$ is not the dominant set, since $\{a_0''\} \not\sim N_1 \cup R_1$ (since $\{a_0''\} \not\sim$N, $N_1 \subseteq N, R_1 \subseteq R$ and $\{a_0''\} \subseteq A$ $\not\sim$R).
I found a connected subgraph, H_0, with the property that no complete bipartite subgraph is dominant. According to Theorem 1 it follows that G is not $\{P_6, C_6\} - free.$ So, N_G(N)-R=A hold.

Theorem 3.1.1. *Let G = (V, E) be a connected and incomplete graph and (A, N, R) a weak decomposition with the G (A) weak component. Let G = (V, E) be a $\{C_3, C_4\}$-free. Then G is $\{P_6, C_6\} - free$ if and only if:*

i) N and R are stable sets;

ii) $N_G(N) - R = A$;

iii) $G(A \cup N)$ is $\{P_6, C_6\} - free$ and $\{C_3, C_4\}$-free;

iv) G(A) is $P_3 - free$.

Proof. Let G be $\{P_6, C_6\} - free.$ From Lemma 3.1.1, it follows that i) and ii) holds. From the property of the heredity of $\{P_6, C_6\} - free$ graphs, it follows that iii) hold.

If G(A) would contain $P_3: a, ab, b, bc, c$ then from Lemma 3.1.1, it follows that: for $a \in A, \exists n_1 \in N$ such that $an_1 \in E;$ for $c \in A, \exists n_2 \in N$ such that $cn_2 \in E.$ We have $n_1 \neq n_2$ since G is C_4-free and $n_1 n_2 \notin E,$ since N is a stable set. So $G(\{a, b, c, n_2, r, n_1\}) \cong C_6,$ ($\forall r \in R$), a contradiction. So, iv) hold.

We suppose i), ii), iii) and iv) holds. We show G is $\{P_6, C_6\} - free.$ From i), it follows that G(N), G(R) are $\{P_6, C_6\} - free.$ From iii), it follows that G(A), G(N) are $\{P_6, C_6\} - free.$ From iii), it follows that $G(A \cup N)$ is $\{P_6, C_6\} - free.$ Since G(A), G(R) are $\{P_6, C_6\} - free$ and A$\not\sim$R, it follows that $G(A \cup R)$ is $\{P_6, C_6\} - free.$ From i), it follows that $G(N \cup R)$ is $\{P_6, C_6\} - free.$

If it existed $X \subseteq V$ with $X \cap A \neq \phi, X \cap N \neq \phi, X \cap R \neq \phi$ such that $G(X) \cong F_6$ or $G(X) \cong C_6$ then would be the following possibilities. Since $N \sim R$ and A$\not\sim$R, it follows that $|X \cap R| = 1$.

The situation $G(X) \cong C_6$ where $C_6 : a, an_1, n_1, n_1r, r, rn_2, n_2, n_2c, c, cb, b, ba$ with $a, b, c \in A$, $n_1, n_2 \in N$, $r \in R$, is not possible since $G(\{a, b, c\}) \cong F_3$, contradicting iv).

The situation $G(X) \cong F_6$ where $F_6 : n_1, n_1r, r, rn_2, n_2, n_2c, c, cb, b, ba, a$ with $a, b, c \in A$, $n_1, n_2 \in N$, $r \in R$, is not possible since $G(\{a, b, c\}) \cong F_3$, contradicting iv).

Let $G(X) \cong F_6$ be where $F_6 : a_1, a_1n_1, n_1, n_1r, r, rn_2, n_2, n_2a_2, a_2, a_2a_3, a_3$ with $a_1, a_2, a_3 \in A$, $n_1, n_2 \in N$, $r \in R$. Since G(A) is connected, it follows that $\exists L_{a_1a_3}$. If $a_1a_3 \in E$ then $G(\{a_1, a_3, a_2\}) \cong F_3$, contradicting iv). If $a_1a_3 \notin E$ then on the path $L_{a_1a_3}$, let a_4 be such that $a_4a_3 \in E$. Then $G(\{a_4, a_3, a_2\}) \cong F_3$, contradicting iv).

Remark 3.1.1. Note that $N_G(N) - R = A$ is used in Lemma (not used (not needed) in Theorem), but is useful in the recognition algorithm.

Remark 3.1.2. In ([12]) we have specified in „Unweighted problems”:

the problem of recognition for $P_3 - free$ is in linear time (see: ([7]).

Remark 3.1.3. In ([13]) we have specified in „Unweighted problems”:

the problem of recognition for $C_4 - free$ is in polynomial time (see: [6]).

1.2. Recognition algorithm for $\{F_6, C_6\} - free$ and $\{C_3, C_4\} - free$ graphs

The characterization theorem leads to the next recognition algorithm.

Recognition algorithm for $\{F_6, C_6\} - free$ and triangle-free graphs

Input: A connected graph with at least two nonadjacent vertices, G=(V,E)

Output: An answer to the question: G is $\{F_6, C_6\} - free$ *and* $\{C_3, C_4\} - free$?

Begin

$L = \{G\} // L$ a list of graphs

while $L \neq \phi$ *do*

extracts an element H from L

determine a weak decomposition (A,N,R) for H with $N \sim R$ //4

if ($\exists a \in A$ such that $\{a\} \not\sim N$) *then*

\ G is not $\{F_6, C_6\} - free$ *//5*

else

if (G(A) is not P_3-free)

then G is not

$\{F_6, C_6\} - free$ *//7*

else

if (N is not stable set or R is not stable set) *then //9*

G is not $\{F_6, C_6\} - free$

else

introduce in *L:*

$G(A \cup N)$ of at

least 6

G is $\{F_6, C_6\} - free$

end

Complexity of the algorithm. Line 4 run O (n + m) time, line 5 run $O(|N|)$ time, line 7 run $O(|A|)$ (result from Remark 3.1.2.), Line 9 run O (m) time (Indeed. In line 4, we each associate adjacent matrices for G (A), G (N), G (R). If these matrices contain 1 for G (N) or G (R), then N or R are not stable sets. So the execution is $O(max\{|N|^2, |E|^2\}) \approx O(m)$.). Execution, in total, is O (n (n + m)) time.

1.3. Combinatorial optimization algorithms for $\{F_6, C_6\} - free$ and $\{C_3, C_4\} - free$ graphs

Next we give combinatorial optimization algorithms of complexity of maximum O (n (n + m)) time.

In ([11]) it is stated that the following problems are executed in polynomial time:

Recognition, Domination are Polynomial.

Proposition 3.3.1. *Let G=(V,E) be a connected graph with at least two nonadjacent vertices and (A,N,R) a weak decomposition with G(A) a weak component. If G is $\{F_6, C_6, C_3, C_4\} - free$ then holds:*

i) N is a minimum dominating set;

ii) $A \cup R$ a dominating set, which is not minimum.

Proof. For $\forall r \in R, \forall n \in N: nr \in E$, since $N \sim R$. For $\forall a \in A$, $\exists n \in N$ such that $an \in E$, since $N_G(N) - R = A$. So N is a dominating set.

If would exist $n_0 \in N$ such that $N - \{n_0\}$ to be dominating set then for $n_0 \notin N - \{n_0\}$ would exist $n' \in N - \{n_0\}$ such that $n_0 n' \in E$, that is N it would not be stable set. So N is minimum dominating set.

We show ii). For $\forall n \in N, \exists r \in R: nr \in E$, since $N \sim R$. So $A \cup R$ is dominating set.

If would exist $r_0 \in R$ such that $A \cup R - \{r_0\}$ to be dominating set then, since A $\not\sim$R, would exist $r_0' \in R - \{r_0\}$ such that $r_0' r_0 \in E$, that is R it would not be stable set.

We show that $\exists a_0 \in A$ such that $A \cup R - \{a_0\}$ to be dominating set. (Indeed. For $a_0 \notin A \cup R - \{a_0\}$, $\exists a_0'' \in A \cup R - \{a_0\}$ such that $a_0'' a_0 \in E$. The vertex $a_0'' \in N_{G(A)}(a_0)$ (the set $N_{G(A)}(a_0) \neq \phi$, since G(A) is connected).

$X \leftarrow A \cup R$

while ($\exists a_0 \in X$ such that $X - \{a_0\}$ is dominating set) *do*

$$X \leftarrow X - \{a_0\}$$

X is a minimum dominating set and it is obtained in $O(|X|) = O(|A| + |R|)$ time.

Corollary 3.3.1. *Let G=(V,E) be a connected graph with at least two nonadjacent vertices and (A,N,R) a weak decomposition with G(A) a weak component. If G is $\{P_6, C_6, C_3, C_4\} - free$ then hold:*

$\gamma(G) = \{|X|, |N|\}$.

Remark 3.3.1. *From Corollary 3.3.1 we obtain the determination of the number of domination in O (n) time.*

CONCLUSIONS

In this paper, using weak decomposition, we characterize *{C_3, C_4}*-free and *{P_6, C_6} -free* graphs. Also, we give a recognition algorithm, and combinatorial optimization algorithms of complexity of maximum *O (n (n + m))* time for this class of graphs.

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