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STABILITY CRITERIA FOR CELLULAR NEURAL NETWORKS

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Abstract: A cellular neural network is an artificial neural network which features a multi-dimensional array of neurons and local interconnections among the cells. Neural networks are systems with several equilibrium states. It is exactly this fact (existence of several equilibria) that grants to the neural networks their computational and problem solving capabilities. The paper is concerned with dynamical properties of the model of a cellular neural network as a dynamical system with several equilibria displaying interaction delays. There are given sufficient conditions for stability of the cellular neural networks.

Keywords: Cellular neural networks, global stability, time delay.

1. INTRODUCTION

Cellular neural networks (CNNs), introduced in 1998 (Chua and Yang, 1998), are artificial neural networks displaying a multidimensional array of cells and local interconnections among the cells. CNNs have been successfully applied to signal processing, image processing, to solve partial differential equations and nonlinear algebraic equations. In such applications stability and other problems of dynamical behaviour of the CNN are equally important. These properties are necessary for the network to achieve its goal and have to be checked on the mathematical model.

Large CNN chips can be implemented using VLSI technology. The finite switching speed of amplifiers and communications time introduce time delays in the signal transmition between the cells. These lags may introduce oscillations or may lead to instability of the network.

In this paper, sufficient conditions for the global stability of a cellular neural network with time delay are stated. These conditions are independent of the delay parameter.

2. THE MATHEMATICAL MODEL AND PROBLEM STATEMENT

The aim of this paper is to obtain sufficient conditions for global stability of a CNN with time lag feedback and zero control template

$$\dot{x}_i = -a_i x_i(t) + \sum_{j \in N} w_{ij} f\left(x_j(t - t_j)\right) + I_i \qquad (1)$$

where j is the index for the cells of the nearest neighbourhood N of the i th cell and a_i is a positive parameter.

First, an isolated cell with their dynamics is considered

$$\dot{x}_i = -a_i x_i(t) + w_{ii} f\left(x_i(t - t_i)\right) + I_i \qquad (2)$$

and stability criteria are stated. Next, we have to consider the interconnections which are nonlinear and to impose the so-called “conditions of passive interconnection” (Willems, 1972) which are sufficient for the exponential stability preservation.

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One can shift the equilibrium point x^* to the origin, so that system (2) can be written into the form:

$$\dot{z}_i = -a_i z_i(t) + w_{ii} g\left(z_i(t-\tau_i)\right) \qquad (3)$$

where $z_i = x_i - x_i^*$ and $g(z_i) = f(z_i + x_i^*) - f(x_i^*)$.

The nonlinearities

$$f(x_i) = \tfrac{1}{2}\left(|x_i + 1| - |x_i - 1|\right) \qquad (4)$$

are bounded, their range being [-1, 1]. Also these functions are monotonically increasing and globally Lipschitzian. This means they satisfy the inequalities

$$0 \le \frac{f(\sigma_1) - f(\sigma_2)}{\sigma_1 - \sigma_2} \le L \qquad (5)$$

or more specific

$$0 \le \frac{f(\sigma)}{\sigma} \le L \qquad (6)$$

since $f(0) = 0$, where the Lipschitz constant is $L = 1$. The function $g(z)$ satisfy the same Lipschitz condition with the same constant $L = 1$. These properties of the nonlinear functions suggest application of the absolute stability theory methods.

The linear shifted subsystem of the isolated cell

$$\begin{aligned} \dot{z}_i &= -a_i z_i(t) + w_{ii} g\left(z_i(t-\tau_i)\right) \\ y_i &= z_i \end{aligned} \qquad (7)$$

has the frequency domain characteristic

$$H_i(j\omega) = \frac{w_{ii}}{a_i + j\omega} e^{-j\omega\tau_i} \qquad (8)$$

3. MAIN RESULT

Theorem: Consider system (7) with frequency domain characteristic (8) under the following assumptions:

i) $a_i > 0$, $i = 1, \ldots, n$;
ii) The nonlinear functions $g(\sigma)$ are globally Lipschitz satisfying

$$0 \le \frac{g(\sigma_1) - g(\sigma_2)}{\sigma_1 - \sigma_2} \le 1, \quad g(0) = 0 \qquad (9)$$

The frequency domain Popov condition

$$1 + \mathrm{Re}\left[(1 + j\omega b) H_i(j\omega)\right] > 0 \qquad (10)$$

holds and the isolated cell has an equilibrium point which is asymptotic stable if the system satisfies one of the following conditions:

a) For excitatory feedback ($w_{ii} > 0$), $w_{ii} \in (0, a_i)$ and $b \in (0, 1/w_{ii})$.
b) For inhibitory feedback ($w_{ii} < 0$), $w_{ii} \in (-a_i, 0)$ and $b \in (-1/w_{ii}, +\infty)$.

The frequency domain condition (10) leads to the following inequality:

$$\begin{aligned} &\omega^2(1 + bw_{ii}\cos\omega\tau_i) + \omega\left[w_{ii}(ba_i - 1)\sin\omega\tau_i\right] \\ &\quad + a_i^2 + w_{ii}a_i\cos\omega\tau_i > 0 \end{aligned} \qquad (11)$$

which holds for

$$A_1 = (1 + bw_{ii}\cos\omega\tau_i) > 0 \qquad (12)$$

and

$$\begin{aligned} \Delta_1 &= w_{ii}^2(ba_i - 1)^2\sin^2\omega\tau_i - \\ &4(1 + bw_{ii}\cos\omega\tau_i)\cdot(a_i^2 + w_{ii}a_i\cos\omega\tau_i) < 0 \end{aligned} \qquad (13)$$

Inequality (13) is equivalent to

$$\begin{aligned} &\left[w_{ii}^2(1 + ba_i)^2\cos^2\omega\tau_i \right.\\ &\quad \left. + 4w_{ii}a_i(1 + a_i b)\cos\omega\tau_i + 4a_i^2\right] - \\ &- w_{ii}^2(1 - ba_i)^2 > 0 \end{aligned} \qquad (14)$$

The left hand side of the inequality (14) is a difference between two squares. We obtain:

$$\begin{aligned} &\left[2a_i + w_{ii}(1 + ba_i)\cos\nu\tau_i - w_{ii}(1 - ba_i)\right]\cdot \\ &\left[2a_i + w_{ii}(1 + ba_i)\cos\nu\tau_i + w_{ii}(1 - ba_i)\right] > 0 \end{aligned} \qquad (15)$$

There are two cases:

1. For excitatory feedback, $w_{ii} > 0$, from (12) it follows $b \in (0, 1/w_{ii})$ for the most unfavourable case $\cos\omega\tau_i = -1$ and from (15) $w_{ii} \in (0, a_i)$ and the same condition $b \in (0, 1/w_{ii})$.
2. For inhibitory feedback, $w_{ii} < 0$, from (12) it follows $b \in (-1/w_{ii}, +\infty)$ for the most unfavourable case $\cos\omega\tau_i = 1$ and from (15) $w_{ii} \in (-a_i, 0)$ and the same condition $b \in (-1/w_{ii}, +\infty)$.

We obtain a delay-independent parameter condition for the asymptotic stability of the equilibrium point of an isolated cell.

The same result may be obtained using the following Liapunov functional defined e.g. on $\mathbf{R} \times \mathbf{L}^2(-\tau_i, 0; \mathbf{R})$ as follows:

$$V(z,\varphi) = \tfrac{1}{2}\alpha_i z^2 + \beta_i \int_{-\tau_i}^{0} \varphi^2(\theta)\,d\theta + \int_0^z g(\sigma)\,d\sigma \tag{16}$$

where $\alpha > 0$, $\beta > 0$ are suitably chosen arbitrary parameters. Along the solutions of the shifted system this functional reads

$$V^*(t) = V(z_i(t), z_{it}) = \tfrac{1}{2}\alpha_i z_i^2(t) + \beta_i \int_{t-\tau_i}^{t} z_i^2(\theta)\,d\theta + \int_0^{z_i(t)} g(\sigma)\,d\sigma \tag{17}$$

Its derivative (along system's solutions) is

$$\begin{aligned}\frac{dV^*}{dt}(t) &= -(\alpha_i a_i - \beta_i) z_i^2(t) - \\ &- \beta_i z_i^2(t-\tau_i) - a_i z_i(t)\, g(z_i(t)) + \\ &+ \alpha_i w_{ii} z_i(t)\, g(z_i(t-\tau_i)) + \\ &+ w_{ii}\, g(z_i(t))\, g(z_i(t-\tau_i))\end{aligned} \tag{18}$$

Taking into account the Lipschitz constant that equals 1, we may use the (rather conservative) estimates:

$$\begin{aligned}&|w_{ii} z_i(t) g(z_i(t-\tau_i))| \le \\ &\le |w_{ii}|\,|z_i(t)|\,|g(z_i(t-\tau_i))| \le \\ &\le |w_{ii}|\,|z_i(t)|\,|z_i(t-\tau_i)|\end{aligned} \tag{19}$$

$$\begin{aligned}&|w_{ii}\, g(z_i(t))\, g(z_i(t-\tau_i))| \le \\ &\le |w_{ii}|\,|z_i(t)|\,|z_i(t-\tau_i)|\end{aligned} \tag{20}$$

and obtain:

$$\begin{aligned}\frac{dV^*}{dt}(t) &\le -(\alpha_i a_i - \beta_i)\, |z_i(t)|^2 - \\ &- \beta_i\, |z_i(t-\tau_i)|^2 + \\ &+ (\alpha_i + 1)\, |w_{ii}|\, |z_i(t)|\, |z_i(t-\tau_i)|\end{aligned} \tag{21}$$

Quadratic form definite sign arguments will give the following choice for α_i and β_i

$$\frac{\alpha_i}{\alpha_i + 1} > \frac{|w_{ii}|}{a_i} \tag{22}$$

$$\begin{aligned}&\frac{\alpha_i}{\alpha_i+1} - \sqrt{\left(\frac{\alpha_i}{\alpha_i+1}\right)^2 - \frac{w_{ii}^{\,2}}{a_i^2}} < \frac{\beta_i}{a_i} < \\ &< \frac{\alpha_i}{\alpha_i+1} + \sqrt{\left(\frac{\alpha_i}{\alpha_i+1}\right)^2 - \frac{w_{ii}^{\,2}}{a_i^2}}\end{aligned} \tag{23}$$

which require as previously the "small gain conditions" $|w_{ii}| < a_i$.

4. INTERCONNECTION EFFECTS

The role of interconnections in stability preserving is crucial. There are several approaches is coping with interconnections (Willems, 1972; Vidyasagar, 1981). All of them are based on the properties of the Liapunov functions and for this reason we took the both approaches in the previous section. Nevertheless there are some additional restrictions. First of them restricts the problem to linear interconnections. This restriction may be overcome by a change of the state variables, which is usual in the theory of neural networks and relies in monotonicity of the nonlinear functions. The second problem is given by the time delay in the interconnections. Such problem occurred previously in the theory of interconnected nuclear reactors (Goriachenko, 1977). Provided Lebesgue or Riemann integrals may be replaced by Stieltjes integrals, this problem is solvable using rearrangement inequalities (Hardy, Littlewood and Polya, 1946).

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