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CONSIDERATIONS ON THE NONLINEARITY GRADE OF THE NONLINEAR MECHANICAL ELASTIC SYSTEMS

Assoc. Prof. Dr. Eng. Nicusor DRAGAN
MECMET - The Research Center of Machines, Mechanic and Technological Equipments
"Dunarea de Jos" University of Galati

ABSTRACT

The article is taking into consideration a nonlinear model for the damping, where the damping coefficient has a polynomial variation function of the velocity. The differential equation of the movements of the non-linear 1DOF system can be solved only using numerical methods. The study gives the physical and mathematical modeling of the dynamics of 1DOF mechanical systems. It also puts forward two quantitative indexes of nonlinearity (the nonlinearity index of spectral amplitudes, the nonlinearity index of spectral power) in order to indicate how big is the nonlinearity grade of the system.

KEYWORDS: nonlinear mechanical system, nonlinearity index

1. INTRODUCTION

The usual dynamics approaches of vibrating machines and equipment consider that the mechanical system (with finite DOF) has discrete components (masses, dampers and elastic springs) with linear behavior. But, there are a lot of situations when the linear / linearized model of the vibrating systems cannot explain some resonance phenomena at the superior or inferior frequencies compared to the driving vibrator frequency or the necessity to supercharge the motor of the vibrator. In this case, a model of the system with nonlinear elasticity and / or damping can lead to some more accurate theoretical results.

Physical and mathematical modeling of linear elastic mechanical systems leads to the second order differential linear equations, with constant coefficients. These equations which model with small enough errors the dynamic behavior of the system are the result of simplifying assumptions involving structural and geometric linearity of the mass / inertia, elasticity and damping.

Nonlinear differential equation of an autonomous 1DOF mechanical system has the general form

$$a(\dot{q},q)\ddot{q}+b(\dot{q},q)\dot{q}+c(\dot{q},q)q=F(t), \qquad (1)$$

where: $q\,/\,\dot{q}\,/\,\ddot{q}$ are generalized coordinate / velocity / acceleration

$a(\dot{q},q)$ - inertial coefficient

$b(\dot{q},q)$ - damping coefficient

$c(\dot{q},q)$ - elasticity coefficient

In most cases, the nonlinear mechanical elastic systems have constant inertial characteristics (mass, moments of inertia), nonlinear behavior being given by dissipative and elastic elements. In general, nonlinearities of elasticity occur in elastic-force strain relationship and the relationship between strain rate and dissipative force resistance element requires linear or nonlinear damping behavior. Under these conditions, nonlinear differential equation system has the form

$$a\ddot{q}+b(\dot{q})\dot{q}+c(q)q=F(t), \qquad (2)$$

damping coefficient being function of speed and stiffness coefficient function of elongation.

For a mechanical elastic 1DOF system with nonlinear damping, the differential equation of forced vibration is as follows:

$$a\ddot{q} + b(\dot{q})\dot{q} + cq = F(t), \tag{3}$$

For the technical and technological mechanical systems, the dissipative nonlinear behavior is determined by the connecting elements made from neoprene, hydraulic and hydro-pneumatic shock absorbers or by the interaction between the work equipment and environment.

2. 1DOF MECHANICAL SYSTEM WITH NONLINEAR DAMPING

Figure 1 shows the simplified model of an inertial vibrator conveyor, with the following notations: **1** – the sieve, **2** – the transporter basis, **3** – the elastic support system (steel bending plates), **4** – the inertial vibrator (m_0 is the total unbalanced mass).

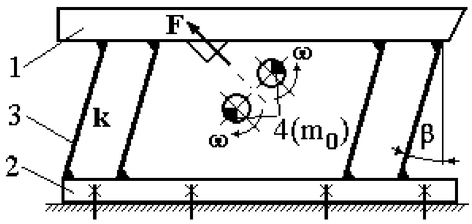

Fig. 1. Simplified model of the inertial vibrating technological equipment

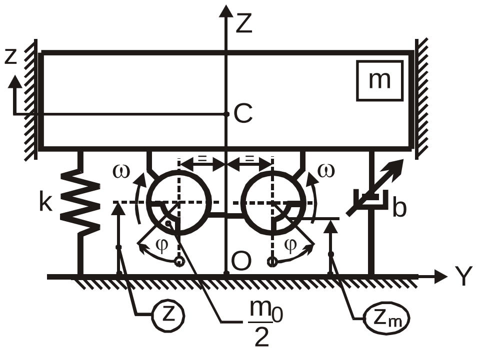

Fig. 2. Mechanical 1DOF model with nonlinear damping of the vibrating equipment

Figure 2 shows the model of the conveyor driven by an inertial vibrator with two eccentric synchronized masses (the model Figure 2 is the vertical plane projection of the real model Figure 1). The used notations are:
C – the mass center of the vibrating system;
m – the total mass of the conveyor (includes the vibrator mass);
m_0 – the total eccentric masses;
k – elasticity coefficient of the conveyor’s steel springs;
b – the dissipation coefficient (that includes the damping of the eaves’ seat and the equivalent dissipation of the transported material);
Z – the vibrating direction;
z – the displacement of the conveyor’s eaves;
z_m – displacement of unbalanced / eccentric masses;
φ – rotation angle of the eccentric masses;
ω – rotation velocity of the eccentric masses.

The measured and the calculated data of the real inertial vibrating conveyor used to numerical simulation are:

- ♦ $m = 250Kg$ - the total vibrating mass of the conveyor (measured);
- ♦ $k = 3 \times 10^5\ Nm^{-1}$ - the coefficient of elasticity of steel springs (measured);
- ♦ $b = 12 \times 10^3\ Nsm^{-1}$ - the equivalent coefficient of dissipation (calculated);
- ♦ $n = 948rpm$ - the rotational speed of eccentric masses (measured);
- ♦ $f = 15.8Hz$ - the frequency of inertial excitation (calculated);
- ♦ $\omega = 99.27rad/s$ - the pulsation of inertial excitation (calculated);
- ♦ $m_0 r = 1.2583Kgm$ - the static moment of the eccentric masses (calculated);
- ♦ $F_0 = 12.4kN$ - the amplitude of one direction inertial force (calculated).

The calculated data of the inertial vibrating conveyor modelled as a linear viscous elastic mechanical system are:

- ♦ $f_n = 5.513Hz$ - the eigenfrequency of the conveyor;
- ♦ $b_{cr} = 17320.5Nsm^{-1}$ - the critical value of the damping coefficient;
- ♦ $n = 24rad/s$ - the damping factor; $\zeta = 0.6928$ - the linear damping ratio;
- ♦ $A_{st} = 5.033mm$ - the steady-state forced vibration amplitude ($A_{st} = \lim_{f \to \infty} A_f$).

3. DYNAMIC ANALYSIS OF THE 1DOF MECHANIC SYSTEM

3.1. Model of the mechanical system with linear damping

Acc. to [1] [3], the steady-state vibrations equations of the conveyor driven by the inertial vibrator are done by

$$\begin{cases} m\ddot{z} + b\dot{z} + kz = m_0 r\omega^2 \cos\omega t \\ M_M = m_0 r(g - \ddot{z})\sin\omega t \end{cases}, \tag{4}$$

where M_M is the necessary motor moment and $g = 9.81 ms^{-2}$.

First eq. from (4) can be written as follows

$$\ddot{z} + 2n\dot{z} + p^2 z = \mu r\omega^2 \cos\varphi, \tag{5}$$

where we have used the notations:

$n = \frac{b}{2m}$ is the damping factor

$p = \sqrt{\frac{k}{m}}$ - the eigenpulsation of the 1DOF linear system

$\mu = \frac{m_0}{m}$ - dimensionless unbalanced mass

$\varphi = \omega t$ - angular displacement of the rotary unbalanced masses.

The forced steady-state vibration of the conveyor is described by the particular solution of eq. (5) as follows

$$z_f = A_f \cos(\omega t - \varphi_0), \tag{6}$$

where the amplitude is

$$A_f = \frac{\mu r\omega^2}{\sqrt{\left(p^2 - \omega^2\right)^2 + 4n^2\omega^2}} \tag{7}$$

and the phase shift between harmonic inertial force and the conveyor vibration is:

$$\varphi_0 = \arctan\frac{2n\omega}{p^2 - \omega^2} \tag{8}$$

From the second eq. of (4), we can write the necessary motor moment M_M as follows:

$$M_M = m_0 r\left[g + A_f\omega^2 \cos(\omega t - \varphi_0)\right]\sin\omega t \tag{9}$$

Taking into consideration the mathematical expressions of the amplitude and phase shift done by the relations (7) and (8), the necessary motor moment M_M becomes:

$$M_M = m_0 rg \sin\omega t + \frac{m_0\mu r^2\omega^4}{2\left[\left(p^2 - \omega^2\right)^2 + 4n^2\omega^2\right]} \cdot \tag{10}$$

$$\cdot\left[\left(p^2 - \omega^2\right)\sin 2\omega t + 2n\omega\left(1 - 2\cos^2\omega t\right)\right]$$

The differential mechanical work of the motor dW can be written

$$dW = M_M d\varphi = M_M \omega dt \tag{11}$$

and the mechanical work for an entire oscillation cycle can be written as follows:

$$W_{cycle} = \int_0^{2\pi} dW = \int_0^{2\pi} M_M d\varphi = \int_0^{\frac{2\pi}{\omega}} M_M \omega dt \tag{12}$$

With the expression (10) of the motor moment M_M, the mechanical work for a cycle becomes after integration as follows:

$$W_{cycle} = \frac{2\pi(m_0 r)^2 n\omega^5}{m\left[\left(p^2 - \omega^2\right)^2 + 4n^2\omega^2\right]} \tag{13}$$

The average necessary motor moment M_{Mavg} and the average power P_{avg} can be calculated as follows:

$$M_{Mavg} = \frac{W_{cycle}}{2\pi} = \frac{(m_0 r)^2 n\omega^5}{m\left[\left(p^2 - \omega^2\right)^2 + 4n^2\omega^2\right]} \tag{14}$$

$$P_{avg} = \omega M_{Mavg} = \frac{(m_0 r)^2 n\omega^6}{m\left[\left(p^2 - \omega^2\right)^2 + 4n^2\omega^2\right]} \tag{15}$$

3.2. Model of the mechanical system with nonlinear damping

In order to make a qualitative and quantitative analysis of the dynamic parameters of the 1DOF mechanical system with nonlinear damping [2] [4], we consider differential moving eq. from (4), where the nonlinear damping coefficient is polynomial type as follows:

$$b = b_0 + \sum_{i=1}^{\infty} b_i |\dot{z}|^i \ , \tag{16}$$

where b_0 is the coefficient of linear damping and $b_i \ \ i = \overline{1,\infty}$ are the coefficients of nonlinear polynomial damping (dissipations proportional to velocity integer exponents).

For qualitative evaluation of the dynamics of the 1DOF system with nonlinear damping, we consider, in the first approximation, that the steady-state vibration is harmonic with the same frequency as the inertial force:

$$z_f = A \cos \omega t \tag{17}$$

The modulus of the velocity can be written as follows

$$\left| \dot{z}_f \right| = A\omega |\sin \omega t| \ , \tag{18}$$

where the module of sine is periodic and we can write it also

$$|\sin \omega t| = \begin{cases} \sin \omega t & if \quad (2k-2)\pi \le \omega t < (2k-1)\pi \\ -\sin \omega t & if \quad (2k-1)\pi \le \omega t < 2k\pi \end{cases} \quad k \in Z \tag{19}$$

Being periodic, the function from (19) can be decomposed into a Fourier series as follows:

$$f(t) = |\sin \omega t| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{i=1}^{\infty} \frac{1}{4i^2 - 1} \cos 2i\omega t \tag{19}$$

Because the coefficients of the harmonic functions rapidly decrease to the *i* index, we consider only the first four terms from the Fourier series as follows:

$$|\sin \omega t| \approx \frac{2}{\pi} - \\ -\frac{4}{\pi}\left(\frac{1}{3}\cos 2\omega t + \frac{1}{15}\cos 4\omega t + \frac{1}{35}\cos 6\omega t \right) \tag{20}$$

With the approximation (20), the expression of the velocity’ modulus becomes

$$\left| \dot{z}_f \right| \approx a_0 + a_2 \cos 2\omega t + \\ + a_4 \cos 4\omega t + a_6 \cos 6\omega t \tag{21}$$

where the coefficients a_{2i}, $i=0,1,2,3$ are as in [3].

Taking into consideration only four terms for the polynomial damping coefficient

$$b \approx b_0 + b_1 \left| \dot{z}_f \right| + b_2 \dot{z}_f^2 + b_3 \left| \dot{z}_f \right|^3 \ , \tag{22}$$

and the modulus of the velocity done by (21), the global damping coefficient can be written as follows:

$$b \approx b_0 + b_1 (a_0 + a_2 \cos 2\omega t + a_4 \cos 4\omega t + \\ + a_6 \cos 6\omega t) + b_2 (a_0 + a_2 \cos 2\omega t + \\ + a_4 \cos 4\omega t + a_6 \cos 6\omega t)^2 + b_3 (a_0 + \\ a_2 \cos 2\omega t + a_4 \cos 4\omega t + a_6 \cos 6\omega t)^3 \tag{23}$$

The square of the modulus of the velocity can be written

$$\dot{z}_f^2 = (a_0 + a_2 \cos 2\omega t + a_4 \cos 4\omega t + \\ + a_6 \cos 6\omega t)^2 = \sum_{i=0}^{6} c_{2i} \cos 2i\omega t \tag{24}$$

where the coefficients $c_{2i} \ \ i = \overline{0,6}$ can be written function of the coefficients a_{2i}, $i=0,1,2,3$, by identification of the coefficients of the trigonometric functions sine and cosine.

The cube of the modulus of the velocity can be written

$$\dot{z}_f^3 = (a_0 + a_2 \cos 2\omega t + a_4 \cos 4\omega t + \\ + a_6 \cos 6\omega t)^3 = \sum_{i=0}^{9} d_{2i} \cos 2i\omega t \tag{25}$$

where the coefficients $d_{2i} \ \ i = \overline{0,9}$ can be written function of the coefficients a_{2i}, $i=0,1,2,3$.

With the expressions (24) and (25) of the exponents of the modulus of the velocity, the damping coefficient (23) becomes

$$b = \sum_{i=0}^{9} e_{2i} \cos 2i\omega t \ , \tag{26}$$

where the coefficients $e_{2i} \ \ i = \overline{0,9}$ can be calculated by a simple identification of the coefficients of the trigonometric functions sine and cosine.

The nonlinear resistance force according to polynomial damping coefficient (26) becomes

$$F_R = -b\dot{z} = A\omega \sin\omega t \sum_{i=0}^{9} e_{2i} \cos 2i\omega t \quad , \qquad (27)$$

or, $$F_R = -\sum_{i=0}^{9} F_{2i+1} \sin(2i+1)\omega t \quad , (28)$$

where: $F_{2i+1} = \dfrac{A\omega}{2}\left(e_{2i} - e_{2i+2}\right) \quad i = \overline{1,8}$

$$F_1 = \frac{A\omega}{2}\left(e_2 - 2e_0\right) \qquad F_{19} = -\frac{A\omega}{2} e_{18}$$

Taking into consideration the nonlinear resistance force done by (28), the differential moving equation becomes

$$m\ddot{z} - F_R + kz = m_0 r\omega^2 \cos\omega t \quad , \qquad (29)$$

or
$$m\ddot{z} + \sum_{i=0}^{9} F_{2i+1} \sin(2i+1)\omega t + kz = m_0 r\omega^2 \cos\omega t \qquad (30)$$

Since we have considered only four terms for the polynomial damping coefficient and four terms for the Fourier series of the modulus of the velocity, the resistance force done by (28) has only ten terms. For an infinite number index the resistance force is as follows:

$$F_R = -\sum_{i=0}^{\infty} F_{2i+1} \sin(2i+1)\omega t = -\sum_{j=1}^{\infty} F_j \sin j\omega t \qquad (31)$$

Taking into consideration only the first $n+1$ (significant) terms of the resistance force, the eq. (30) becomes as follows:

$$m\ddot{z} + F_1 \sin\omega t + kz = m_0 r\omega^2 \cos\omega t - F_3 \sin 3\omega t - F_5 \sin 5\omega t - \ldots - F_{2n+1} \sin(2n+1)\omega t \qquad (32)$$

It can be seen that the right side of the eq. (32) contains not only the harmonic force with the pulsation ω (due to the inertial vibratory) but also harmonic forces with pulsations $(2i+1)\omega \quad i = \overline{1,n}$; that’s why, we can say that the mechanical elastic system with polynomial dissipation excited by harmonic forces is self excited on the odd index superior harmonic frequencies / pulsations.

4. NONLINEARITY INDEX

Considering for the 1DOF mechanical system with polynomial damping the differential equation (32), the forced steady-state motion is composed from the harmonic vibration

$$z(t) = \sum_{i=0}^{n} A_{f,2i+1} \sin\left[(2i+1)\omega t - \varphi_{2i+1}\right] \; , \quad (33)$$

where $A_{f,2i+1} \quad i = \overline{0,n}$ are the spectral harmonic amplitudes of the steady-state vibration and $\varphi_{2i+1} \quad i = \overline{0,n}$ are the phase shifts between the harmonic inertial force and the spectral vibration.

4.1. Nonlinearity index of spectral amplitudes

In order to appreciate the nonlinearity of a mechanical system with polyharmonic steady-state vibrating movement, we can compare the amplitude of the vibration on fundamental pulsation ω with the amplitudes of the vibration on superior spectral pulsations $(2i+1)\omega \quad i = \overline{1,n}$; for this comparison we introduce the nonlinearity index of amplitude defined as follows:

$$I_{A,2i+1} = 100 \frac{A_{f,2i+1}}{A_{f1}} \quad [\%] \quad i = \overline{1,n} \; , \quad (34)$$

where $I_{A,2i+1} \quad i = \overline{1,n}$ is the nonlinearity index of spectral amplitude of $2i+1$ order.

4.2. Nonlinearity index of spectral power

In order to highlight how the power influences the degree of nonlinearity of the system, we can write the mechanical work of the motor for a complete period $T = 2\pi/\omega$ function of forced steady/state vibration amplitude as follows:

$$W_{cycle} = \int_0^{2\pi} M_M d\varphi = \int_0^{2\pi} m_0 r\left[g + A_f \omega^2 \cos(\varphi - \varphi_0)\right] \sin\varphi d\varphi \qquad (35)$$

After the calculus of the definite integrale, the mechanical work becomes

$$W_{cycle} = \pi m_0 r A_f \omega^2 \sin\varphi_0 \qquad (36)$$

or, taking into consideration the expression (8):

$$W_{cycle} = \frac{2\pi(m_0 r)nA_f\omega^3}{\sqrt{\left(p^2-\omega^2\right)^2+4n^2\omega^2}} \quad (37)$$

The average power can be obtained as follows:

$$P_{avg} = \frac{W_{cycle}}{\frac{2\pi}{\omega}} = \frac{(m_0 r)nA_f\omega^4}{\sqrt{\left(p^2-\omega^2\right)^2+4n^2\omega^2}} \quad (38)$$

For the steady-state vibration of the mechanical systems with polynomial damping, the average spectral powers can be written function of spectral amplitudes $A_{f,2i+1}$ and spectral damping factor n_{2i+1} as follows

$$P_{2i+1avg} = \frac{A}{\sqrt{B}} , \quad (39)$$

where:

$$A = (m_0 r)n_{2i+1}A_{f,2i+1}[(2i+1)\omega]^4 , \; i=\overline{1,n}$$

$$B = \left\{p^2 - [(2i+1)\omega]^2\right\}^2 + 4n_{2i+1}^2[(2i+1)\omega]^2$$

We can compare the spectral powers dividing each of them by the fundamental pulsation power as follows:

$$\frac{P_{2i+1avg}}{P_{1avg}} = (2i+1)^4\frac{C}{D} , \quad (40)$$

where $D = n_1 A_{f1}\sqrt{B}$ for $i=\overline{1,n}$.

$$C = n_{2i+1}\cdot A_{f,2i+1}\cdot\sqrt{\left(p^2-\omega^2\right)^2+4n_1^2\omega^2}$$

If we consider that the vibratory technological equipment usually works far off resonance, the relation (40) becomes:

$$\frac{P_{2i+1avg}}{P_{1avg}} \approx (2i+1)^3\cdot\frac{n_{2i+1}A_{f,2i+1}}{n_1 A_{f1}}\cdot E , \quad (41)$$

where $E = \sqrt{\frac{\omega^2+4n_1^2}{[(2i+1)\omega]^2+4n_{2i+1}^2}}$, $i=\overline{1,n}$.

Considering that the square of the spectral pulsations are much bigger than the square of the spectral damping factors, the fraction (41) becomes more simple:

$$\frac{P_{2i+1avg}}{P_{1avg}} \approx (2i+1)^2\frac{n_{2i+1}A_{f,2i+1}}{n_1 A_{f1}} \quad i=\overline{1,n} , \quad (42)$$

In the case of spectral damping factors with close values, we can write:

$$\frac{P_{2i+1avg}}{P_{1avg}} \approx (2i+1)^2\frac{A_{f,2i+1}}{A_{f1}} \quad i=\overline{1,n} \quad (43)$$

The nonlinearity index of spectral power is defined as the ratio between the dissipated powers as follows:

$$I_{P,2i+1} = 100\frac{P_{2i+1avg}}{P_{1avg}} \; [\%] \quad i=\overline{1,n} \quad (44)$$

The relations between the two indexes (34) and (44) are as follows:

$$I_{P,2i+1} = (2i+1)^2\cdot I_{A,2i+1} \quad i=\overline{1,n} \quad (47)$$

5. CONCLUSIONS

The nonlinear mechanical elastic system with polynomial dissipation excited by harmonic forces is self excited on the odd index superior harmonic frequencies / pulsations.

The defined nonlinearity indexes can give a quantitative estimate of the size of the nonlinearity of the system. These indexes can be calculated only after a spectral analysis of the mechanical system vibration is done.

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