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Mean value theorem for a real continuous function with several real variables

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Abstract
In this paper, we establish a mean value theorem for a real continuous function with several real variables, using the Frechet semi-differentials of the function.

1. INTRODUCTION

Given a function $f: X \to R,$ differentiable, where X is a real normed space, the classical mean value theorem states that for $a, b \in X$ there exists c between a and b such that

$$f(b) - f(a) = < Df(c), b - a > .$$

Many authors have obtained various forms of the mean value theorem in the non-smooth case, where f is a convex function ([4]) or a lipschitzian function (see egg. [1] or [5]).
In this paper, we establish a mean value theorem for a continuous function $f : R^n \to R,\ n \geq 2,$ starting from a result obtained in [3] for a real continuous function $f : R \to R,$ using the Frechet semi-differentials of the function.

Let $\Omega \subset R^n$ be an open subset and $x \in \Omega.$ We recall that $f : \Omega \to R$ is Frechet differentiable at x if there exists $\xi \in R^n$ (denoted by $\xi = Df(x)$) such that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - < \xi, h >}{\|h\|} = 0 \qquad (1)$$

One of the definitions of the Frechet semi-differentials is the fact that (1) is equivalent on the conditions that:

$$\limsup_{h \to 0} \frac{f(x+h) - f(x) - < \xi, h >}{\|h\|} \leq 0$$

and

$$\liminf_{h\to 0} \frac{f(x+h)-f(x)-<\xi,h>}{\|h\|} \geq 0.$$

Definition 1. *Let* $\Omega \subset R^n$ *be open,* $f:\Omega \to R$ *and* $x \in \Omega$. *The (possibly empty) subset* $\partial_F^+ f(x) \subset R^n$ *defined by*

$$\partial_F^+ f(x) = \{\xi \in R^n : \limsup_{h\to 0} \frac{f(x+h)-f(x)-<\xi,h>}{\|h\|} \leq 0\},$$

is said to be the Frechet super-differential of f at x and the (possibly empty) subset

$\partial_F^- f(x) \subset R^n$ *defined by*

$$\partial_F^- f(x) = \{\xi \in R^n : \liminf_{h\to 0} \frac{f(x+h)-f(x)-<\xi,h>}{\|h\|} \geq 0\},$$

is said to be the Frechet sub-differential of f at x.

Now we define $\partial_F f(x) = \partial_F^+ f(x) \cup \partial_F^- f(x)$ the union of Frechet semi-differentials of f at *x*.

The basic properties of the Frechet semi-differential are presented in the following propositions (see egg. [2] or [6]):

Proposition 1.1. *If* $x \in \Omega$ *is a local maximum point for* $f:\Omega \to R$ *then* $0 \in \partial_F^+ f(x)$ *and if x is a local minimum point for f then* $0 \in \partial_F^- f(x)$.

Proposition 1.2. *If* $\Omega \subset R^n$ *is open,* $f:\Omega \to R$ *is a continuous function and* $g:\Omega \to R$ *is differentiable at x then:*

$$\partial_F^+ (f+g)(x) = \partial_F^+ f(x) + \partial_F^+ g(x)$$

$$\partial_F^- (f+g)(x) = \partial_F^- f(x) + \partial_F^- g(x).$$

Proposition 1.3. *Let* $\Omega \subset R^n$ *be open,* $f:\Omega \to R^n$, $X \subset R^n$ *be open such that* $f(\Omega) \subset X$ *and* $g:\Omega \to R$ *be a continuous function. If* f *is a local* C^1*- diffeomorfism on* Ω, *then for any* $x \in \Omega$:

$$\partial_F^+ (g \circ f)(x) = \partial_F^+ (g(f(x)) \cdot Df(x)$$

$$\partial_F^-(g \circ f)(x) = \partial_F^-(g(f(x)) \cdot Df(x).$$

The following result was obtained by Cringanu ([3]):

Theorem 1.4. *Let $f : R \to R$ be a continuous function and $a, b \in R$, $a < b$. Then there exists $c \in (a,b)$ such that*

$$f(b) - f(a) = < \partial_F f(c), b - a >$$

so there exists $\alpha \in \partial_F f(c)$ such that

$$f(b) - f(a) = < \alpha, b - a > .$$

2. THE MAIN RESULT

Theorem 2.1. *Let $f : R^n \to R$ be a continuous function, $a, b \in R^n$, $a = (a_1, a_2, ...a_n)$, $b = (b_1, b_2, ...b_n)$ and the following partial functions of f:*

$f_1(t) = f(t, b_2, ...b_n)$, $f_2(t) = f(a_1, t, b_3, ...b_n)$,..., $f_n(t) = f(a_1, a_2, ...a_{n-1}, t)$.

Then there exists $c = (c_1, c_2, ...c_n)$, c_k between a_k and b_k, $1 \le k \le n$, such that

$$f(b) - f(a) \in < \prod_{k=1}^{n} \partial_F f_k(c_k), b - a >$$

so there exists

$$\alpha \in \prod_{k=1}^{n} \partial_F f_k(c_k), \alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \quad \text{with } \alpha_k \in \partial_F f_k(c_k)$$

such that

$$f(b) - f(a) = < \alpha, b - a > = \sum_{k=1}^{n} \alpha_k (b_k - a_k).$$

Proof.

$$f(b) - f(a) = f(b_1, b_2, ..., b_n) - f(a_1, a_2, ..., a_n) = [f(b_1, b_2, ..., b_n) - f(a_1, b_2, ..., b_n)] +$$

$$+ [f(a_1, b_2, ..., b_n) - f(a_1, a_2, ..., b_n)] + [f(a_1, a_2, ..., b_n) - f(a_1, a_2, ..., a_n)] =$$

$$= [f_1(b_1) - f_1(a_1)] + [f_2(b_2) - f_2(a_2)] + ... + [f_n(b_n) - f_n(a_n)].$$

By the theorem 1.4 there exists c_k between a_k and b_k, $1 \le k \le n$, such that

$$f_k(b_k) - f_k(a_k) \in < \partial_F f_k(c_k), b_k - a_k >$$

so there exists $\alpha_k \in \partial_F f_k(c_k)$ such that

$$f_k(b_k) - f_k(a_k) = \alpha_k (b_k - a_k) .$$

Taking $c = (c_1, c_2, ...c_n)$ and $\alpha = (\alpha_1, \alpha_2, ...\alpha_n),$ we get

$$f(b) - f(a) = \sum_{k=1}^{n} \alpha_k (b_k - a_k) = < \alpha, b - a > .$$

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