An implemented algorithm for computing the distributive lattice associated to a Cohen-Macaulay bipartite graph

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Abstract
We give an implemented algorithm for computing the distributive lattice associated to a Cohen-Macaulay bipartite graph and, consequently, the set of all minimal vertex covers of a Cohen-Macaulay bipartite graph

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1. INTRODUCTION

In the first part of the paper we introduce the definitions and the concepts that we operate with and we fix the notation exactly as we did it in [3] and [4].

Let $G = (V, E)$ be a simple (i.e. finite, undirected, loopless and without multiple edges) graph with vertex set $V = [n]$ and the edge set $E = E(G)$. A vertex cover of $G$ is a subset $C \subseteq V$ such that $C \cap \{i, j\} \neq \emptyset$, for any edge $\{i, j\} \in E(G)$. A vertex cover $C$ of $G$ is called minimal if no proper subset $C' \subset C$ is a vertex cover of $G$. A graph $G$ is called unmixed if all minimal vertex covers of $G$ have the same cardinality. Let $R = K[x_1, x_2, \ldots, x_n]$ be the polynomial ring in $n$ variables over a field $K$. The edge ideal of $G$ is the monomial ideal $(I(G))$ of $R$ generated by all the quadratic monomials $x_i x_j$ with $\{i, j\} \in E(G)$. It is said that a graph $G$ is Cohen-Macaulay (over $K$) if the quotient ring $R/I(G)$ is Cohen-Macaulay. Every Cohen-Macaulay graph is unmixed.

Let $P_n = \{p_1, p_2, \ldots, p_n\}$ be a poset with partial order $\leq$. Let $G = G(P_n)$ be the bipartite graph on the set $V_n = W \cup W'$, where $W = \{x_1, x_2, \ldots, x_n\}$ and $W' = \{y_1, y_2, \ldots, y_n\}$, whose edge set $E(G)$ consists of all 2-element subsets $\{x_i, y_j\}$ with $p_i \leq p_j$. It is said that a bipartite graph on $V_n = W \cup W'$ comes from a poset, if there is a finite poset $P_n$ on $\{p_1, p_2, \ldots, p_n\}$ such that $p_i \leq p_j$ implies $i \leq j$ and after relabeling of the vertices of $G$ one has $G = G(P_n)$.

Herzog and Hibi proved in [1, Theorem 3.4] that a bipartite graph $G$ is Cohen-Macaulay if and only if $G$ comes from a poset.

We denote by $M(G)$ the set of all minimal vertex covers of $G$.

Let $L_n$ be the Boolean lattice on the set $\{p_1, p_2, \ldots, p_n\}$. We consider the subset:

$L_\alpha = \{\alpha \subset \{p_1, p_2, \ldots, p_n\} \mid (\exists) C \in M(G): p_i \in \alpha \iff x_i \in C \subseteq L_n\}$. 


A subset \( \alpha \subseteq P_n \) is called a post ideal of \( P_n \) if for every \( a \in \alpha \) and \( b \in P_n, b \leq a \) implies \( b \in \alpha \). We denote by \( I(P_n) \) the set of all poset ideals of \( P_n \). If \( \alpha, \beta \in I(P_n) \), then \( \alpha \cap \beta \in P_n \) and \( \alpha \cup \beta \in P_n \), hence, \( I(P_n) \) is a lattice ordered by inclusion. Moreover, \( I(P_n) \) is a distributive lattice.

Herzog, Hibi and Ohsugi proved in [2] that there is a one-to-one correspondence between the set \( M(G) \) of all minimal vertex covers of \( G \) and the distributive lattice \( I(P_n) \) of all poset ideals of \( P_n \).

Our aim in this paper is to give an implemented algorithm for computing the distributive lattice associated to a Cohen-Macaulay bipartite graph and, consequently, the set of all minimal vertex covers of a Cohen-Macaulay bipartite graph.

2. THE DISTRIBUTIVE LATTICE ASSOCIATED TO A COHEN-MACAULEY BIPARTITE GRAPH AND THE MINIMAL VERTEX COVERS OF A COHEN-MACAULEY BIPARTITE GRAPH

In this section we focus mainly from the computational point of view on the set of all minimal vertex covers of a Cohen-Macaulay bipartite graph, a particular case of an unmixed bipartite graph, for computing the set of all minimal vertex covers of a Cohen-Macaulay bipartite graph.

Proposition 2.1. [4] Let \( P_n = \{p_1, p_2, \ldots, p_n\} \), \( n \geq 2 \), be a poset such that \( p_i \leq p_j \) implies \( i \leq j \) and let \( G_n = G(P_n) \) be the graph on \( V_n \) that comes from the poset \( P_n \). Let \( G_{n-1} \) be the subgraph of \( G_n \) induced by the subset \( V_{n-1} = V_n \setminus \{x_n, y_n\} \). Then a subset \( C_n \subseteq V_n \) is a minimal vertex cover of \( G_n \) if and only if either \( C_n = C_{n-1} \cup \{y_n\} \), where \( C_{n-1} \subseteq V_{n-1} \) is a minimal vertex cover of \( G_{n-1} \) or \( C_n = C_{n-1} \cup \{x_n\} \), where \( C_{n-1} \subseteq V_{n-1} \) is a minimal vertex cover of \( G_{n-1} \) such that \( x_i \in C_{n-1} \), for each \( i \in \{n-1\} \) with \( p_i < p_j \).

As an application of the Proposition 2.1, we can recover a result on the lattice associated to a Cohen-Macaulay bipartite graph ([2, Lemma 2.1]).

Corollary 2.2. [2,4] Let \( G_n = G(P_n) \), where \( P_n = \{p_1, p_2, \ldots, p_n\} \) is a poset such that \( p_i \leq p_j \) implies \( i \leq j \). Then \( L_{G_n} = I(P_n) \).

Next we present our algorithm that was given in [4] for computing the distributive lattice \( I(P_n) \) and, consequently, the set \( M(G) \) of all minimal vertex covers of \( G \), where \( G \) is a bipartite graph that comes from a poset \( P_n = \{p_1, p_2, \ldots, p_n\} \) such that \( p_i \leq p_j \) implies \( i \leq j \).

Algorithm 2.3. [4] The algorithm, based on recursion, computes the lattice \( I(P_k) \), for all \( k \in [n] \), where \( P_k \) is the subposet of \( P_n \) induced by the subset \( \{p_1, p_2, \ldots, p_k\} \). It starts with \( I(P_1) = \{\emptyset, \{p_1\}\} \). At each step \( k \), \( 2 \leq k \leq n \), let us assume that the lattice \( I(P_{k-1}) \) has already been computed. We consider the set \( I_k = \{p_j \mid p_j < p_k, 1 \leq j \leq k - 1\} \). By the Proposition 2.1 and the Corollary 2.2, \( I(P_k) = I(P_{k-1}) \cup I_k \), where \( I_k = \{\alpha \cup \{p_k\} \mid \alpha \in I(P_{k-1}), L(p_k) \subseteq \alpha\} \).

In the final part of the algorithm we compute all minimal vertex covers \( C_\alpha \) of \( G \) by the corresponding poset ideal \( \alpha \) of \( P_n \).

Input: a poset \( P_n = \{p_1, p_2, \ldots, p_n\} \) such that \( p_i \leq p_j \) implies \( i \leq j \)

Output: the lattice \( I \) of all poset ideals of \( P_n \) and the set \( M \) of all minimal vertex covers of \( G(P_n) \)
\[ I = \{\emptyset, \{p_1\}\} \quad \text{(Initially,} \quad I(P_1) = \{\emptyset, \{p_1\}\} \}\]

for \( k = 2, n \) do
\[ L = \emptyset \quad \text{\{We compute the set} \quad L(p_k) = \{p_j \mid p_j < p_k, 1 \leq j \leq k - 1\} \}\]

for \( j = 1, k - 1 \) do
if \( p_j < p_k \) then
\[ L = L \cup \{p_j\} \]
end if
end for

\[ I' = \emptyset \quad \text{\{We compute the set} \quad I'_k = \{\alpha \cup \{p_k\} \mid \alpha \in I(P_{k-1}), L(p_k) \subseteq \alpha\} \}\]

for all \( \alpha \in I \) do
if \( L \subseteq \alpha \) then
\[ I' = I' \cup \{\alpha \cup \{p_k\}\} \]
end if
end for

\[ I = I \cup I' \quad \text{\{We compute the lattice} \quad I(P_k) = I(P_{k-1}) \cup I'_k \}\]
end for

\[ M = \emptyset \quad \text{\{We compute the set} \quad M \text{ by the lattice} \quad I ; \text{ initially,} \quad M = \emptyset \}\]

for all \( \alpha \in I \) do
\[ C = \emptyset \quad \text{\{We compute} \quad C_\alpha = \{x_j \mid p_j \in \alpha\} \cup \{y_j \mid p_j \in \alpha\} \}\]

for \( j = 1, n \) do
if \( p_j \in \alpha \) then
\[ C = C \cup \{x_j\} \]
else
\[ C = C \cup \{y_j\} \]
end if
end for

\[ M = M \cup \{C\} \quad \text{\{Each time we get a minimal vertex cover} \quad C_\alpha , \text{ we add it to} \quad M \}\]
end for

We give an implementation of the previous algorithm in Turbo Pascal Version 7.0.

```pascal
program Lattice_associated_to_a_Cohen_Macaulay_bipartite_graph;
uses Crt;

type
  vector = array[1..100] of integer;
  relation = array[1..100, 1..100] of integer;
  m_int = set of 0..10;

var
  i,j,k,l,m,n: integer;
  level: vector;
  R: relation;
  ML: m_int;
  IP: array[1..10, 1..1024] of m_int;
begin
  ClrScr;
  writeln('Introduce the cardinality of the poset')
```

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write('n=');readln(n);
writeln(n);
for i:=1 to n do
  for j:=1 to n do
  begin
    if i=j then R[i,j]:=1
    else R[i,j]:=0;
  end;
writeln('introduce the number of the relations of type p[i]<p[j] implies i<j');
write('m=');readln(m);
writeln('introduce the relations of type p[i]<p[j] implies i<j');
for k:=1 to m do
  begin
    write('i=');readln(i);
    write('j=');readln(j);writeln;
    R[i,j]:=1;
  end;
writeln('Write all the relations of the poset);
for i:=1 to n do writeln('p[',i,']=p[',i,']');
for i:=1 to n-1 do
  begin
    for j:=i+1 to n do
      if R[i,j]=1 then writeln('p[',i,']=p[',j,']');
  end;
k:=1;
l:=1;
IP[k,l]=[];
l:=l+1;
IP[k,l]=[1];
level[k]:=l;
if n>1 then
  begin
    repeat
      k:=k+1;
      level[k]:=level[k-1];
      for l:=1 to level[k] do IP[k,l]:=IP[k-1,l];
      ML:=[];
      for j:=1 to level[k-1] do
        if R[j,k]=1 then ML:=ML+[j];
      for l:=1 to level[k-1] do
        if ML<=IP[k-1,l] then
          begin
            IP[k,level[k]+1]:=IP[k-1,l]+[k];
            level[k]:=level[k]+1;
          end;
    until k=n;
  end;
for t:=1 to level[n] do
  begin
    write('Poset ideal ','t,':');
    if IP[n,t]=[] then writeln('empty set')
    else
      begin
        for i:=1 to n dp
  end;
if I in IP[n,t] then write('p[',i,','] ');writeln;
end;
end;
readln;
writeln;
for t:=1 to level[n] do
  begin
    write('Minimal vertex cover number ',t,':');
    if IP[n,t]=[] then
      for i:=1 to n do write('y[',i,','] ')
    else
      begin
        for i:=1 to n do
          if i in IP[n,t] then write('x[',i,','] ')
        else write('y[',i,','] ');
      end;
    writeln;
  end;
end.

Example 2.4. Let $P_4 = \{p_1, p_2, p_3, p_4\}$ be the poset with $p_1 \leq p_2$, $p_1 \leq p_4$ and $p_2 \leq p_3$. The Hasse diagram of $P_4$ is represented in the next figure:

![Hasse diagram](image_url)

Let $G = G(P_4)$ be the bipartite graph that comes from the poset $P_4$. The graph is represented geometrically in the next figure:
By using the program based on the Algorithm 2.3, we get the distributive lattice associated to the graph \( G \):
\[
I(P_3) = \{\emptyset, \{p_1\}, \{p_1, p_2\}, \{p_1, p_2, p_3\}, \{p_1, p_2, p_4\}, \{p_1, p_2, p_3, p_4\}\}.
\]
The Hasse diagram of the poset \((I(P_3), \subseteq)\) is depicted in the next figure:

Finally, we get the set \( M(G) \) of all minimal vertex covers of the graph \( G \):
\[
M(G) = \{\{y_1, y_2, y_3, y_4\}, \{x_1, x_2, y_3, y_4\}, \{x_1, x_2, x_3, y_4\}, \{x_1, x_2, x_3, y_3\}, \{x_1, x_2, y_3, y_4\}, \{x_1, x_2, y_2, y_3\}, \{x_1, x_2, y_2, y_4\}, \{x_1, x_2, x_3, y_4\}\}.
\]
References