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An implemented algorithm for computing the distributive lattice associated to a Cohen-Macaulay bipartite graph

Cristian Ion

“Dunarea de Jos” University of Galati, Faculty of Sciences and Environment,
111 Domneasca street, 800201 Galati, Romania
Corresponding author: cristian.adrian.ion@gmail.com

Abstract
We give an implemented algorithm for computing the distributive lattice associated to a Cohen-Macaulay bipartite graph and, consequently, the set of all minimal vertex covers of a Cohen-Macaulay bipartite graph
Keywords: Cohen-Macaulay bipartite graph, vertex cover, distributive lattice, algorithm.

1. INTRODUCTION

In the first part of the paper we introduce the definitions and the concepts that we operate with and we fix the notation exactly as we did it in [3] and [4].

Let $G=(V,E)$ be a simple (i.e. finite, undirected, loopless and without multiple edges) graph with vertex set $V=[n]$ and the edge set $E=E(G)$. A *vertex cover* of G is a subset $C\subset V$ such that $C\cap\{i,j\}\neq\emptyset$, for any edge $\{i,j\}\in E(G)$. A vertex cover C of G is called *minimal* if no proper subset $C'\subset C$ is a vertex cover of G. A graph G is called *unmixed* if all minimal vertex covers of G have the same cardinality. Let $R=K[x_1,x_2,...,x_n]$ be the polynomial ring in n variables over a field K. The *edge ideal* of G is the monomial ideal $I(G)$ of R generated by all the quadratic monomials x_ix_j with $\{i,j\}\in E(G)$. It is said that a graph G is *Cohen-Macaulay* (over K) if the quotient ring $R/I(G)$ is Cohen-Macaulay. Every Cohen-Macaulay graph is unmixed.

Let $P_n=\{p_1,p_2,...,p_n\}$ be a poset with partial order $\leq$. Let $G=G(P_n)$ be the bipartite graph on the set $V_n=W\cup W'$, where $W=\{x_1,x_2,...,x_n\}$ and $W'=\{y_1,y_2,...,y_n\}$, whose edge set $E(G)$ consists of all 2-element subsets $\{x_i,y_j\}$ with $p_i\leq p_j$. It is said that a bipartite graph on $V_n=W\cup W'$ *comes from a poset*, if there is a finite poset P_n on $\{p_1,p_2,...,p_n\}$ such that $p_i\leq p_j$ implies $i\leq j$ and after relabeling of the vertices of G one has $G=G(P_n)$.

Herzog and Hibi proved in [1, Theorem 3.4] that a bipartite graph G is Cohen-Macaulay if and only if G comes from a poset.

We denote by $M(G)$ the set of all minimal vertex covers of G.

Let L_n be the Boolean lattice on the set $\{p_1,p_2,...,p_n\}$. We consider the subset:

$$L_G=\{\alpha\subset\{p_1,p_2,...,p_n\}\mid(\exists)C\in M(G):\ p_i\in\alpha\Leftrightarrow x_i\in C\}\subset L_n.$$

A subset $\alpha \subset P_n$ is called a *poset ideal* of P_n if for every $a \in \alpha$ and $b \in P_n, b \le a$ implies $b \in \alpha$. We denote by $I(P_n)$ the set of all poset ideals of P_n. If $\alpha, \beta \in I(P_n)$, then $\alpha \cap \beta \in P_n$ and $\alpha \cup \beta \in P_n$, hence, $I(P_n)$ is a lattice ordered by inclusion. Moreover, $I(P_n)$ is a distributive lattice.

Herzog, Hibi and Ohsugi proved in [2] that there is a one-to-one correspondence between the set $M(G)$ of all minimal vertex covers of G and the distributive lattice $I(P_n)$ of all poset ideals of P_n.

Our aim in this paper is to give an implemented algorithm for computing the distributive lattice associated to a Cohen-Macaulay bipartite graph and, consequently, the set of all minimal vertex covers of a Cohen-Macaulay bipartite graph.

2. THE DISTRIBUTIVE LATTICE ASSOCIATED TO A COHEN-MACAULEY BIPARTITE GRAPH AND THE MINIMAL VERTEX COVERS OF A COHEN-MACAULAY BIPARTITE GRAPH

In this section we focus mainly from the computational point of view on the set of all minimal vertex covers of a Cohen-Macaulay bipartite graph, a particular case of an unmixed bipartite graph, and, consequently, on the distributive lattice associated to a Cohen-Macaulay bipartite graph.

Firstly, let us present our result, inspired by [5, Lemma 2.5], which provides a recursive method for computing the set of all minimal vertex covers of a Cohen-Macaulay bipartite graph.

Proposition 2.1.([4]) Let $P_n = \{p_1, p_2, ..., p_n\}$, $n \ge 2$, be a poset such that $p_i \le p_j$ implies $i \le j$ and let $G_n = G(P_n)$ be the graph on V_n that comes from the poset P_n. Let G_{n-1} be the subgraph of G_n induced by the subset $V_{n-1} = V_n \setminus \{x_n, y_n\}$. Then a subset $C_n \subset V_n$ is a minimal vertex cover of G_n if and only if either $C_n = C_{n-1} \cup \{y_n\}$, where $C_{n-1} \subset V_{n-1}$ is a minimal vertex cover of G_{n-1} or $C_n = C_{n-1} \cup \{x_n\}$, where $C_{n-1} \subset V_{n-1}$ is a minimal vertex cover of G_{n-1} such that $x_i \in C_{n-1}$, for each $i \in [n-1]$ with $p_i < p_j$.

As an application of the Proposition 2.1. we can recover a result on the lattice associated to a Cohen-Macaulay bipartite graph ([2, Lemma 2.1.]).

Corollary 2.2.([2,4]) Let $G_n = G(P_n)$, where $P_n = \{p_1, p_2, ..., p_n\}$ is a poset such that $p_1 \le p_j$ implies $i \le j$. Then $L_{G_n} = I(P_n)$.

Next we present our algorithm that was given in [4] for computing the distributive lattice $I(P_n)$ and, consequently, the set $M(G)$ of all minimal vertex covers of G, where G is a bipartite graph that comes from a poset $P_n = \{p_1, p_2, ..., p_n\}$ such that $p_i \le p_j$ implies $i \le j$.

Algorithm 2.3.([4]) The algorithm, based on recursion, computes the lattice $I(P_k)$, for all $k \in [n]$, where P_k is the subposet of P_n induced by the subset $\{p_1, p_2, ..., p_k\}$. It starts with $I(P_1) = \{\emptyset, \{p_1\}\}$. At each step k, $2 \le k \le n$, let us assume that the lattice $I(P_{k-1})$ has already been computed. We consider the set $L(p_k) = \{p_j \mid p_j < p_k, 1 \le j \le k-1\}$. By the Proposition 2.1. and the Corollary 2.2. $I(P_k) = I(P_{k-1}) \bigcup I_k^{'}$, where $I_k^{'} = \{\alpha \cup \{p_k\} \mid \alpha \in I(P_{k-1}), L(p_k) \subset \alpha\}$.

In the final part of the algorithm we cojmpute all minimal vertex covers C_α of G by the corresponding poset ideal α of P_n.

Input: a poset $P_n = \{p_1, p_2, ..., p_n\}$ such that $p_i \le p_j$ implies $i \le j$

Output: the lattice I of all poset ideals of P_n and the set M of all minimal vertex covers of $G(P_n)$

$I = \{\varnothing, \{p_1\}\}$ {Initially, $I(P_1) = \{\varnothing, \{p_1\}\}$}
for $k = 2, n$ do
$L = \varnothing$ {We compute the set $L(p_k) = \{p_j \mid p_j < p_k, 1 \le j \le k-1\}$}
for $j = 1, k-1$ do
if $p_j < p_k$ then
$L = L \bigcup \{p_j\}$
end if
end for
$I' = \varnothing$ {We compute the set $I'_k = \{\alpha \cup \{p_k\} \mid \alpha \in I(P_{k-1}), L(p_k) \subset \alpha\}$}
for all $\alpha \in I$ do
if $L \subset \alpha$ then
$I' = I' \bigcup \{\alpha \cup \{p_k\}\}$
end if
end for
$I = I \bigcup I'$ {We compute the lattice $I(P_k) = I(P_{k-1}) \bigcup I'_k$}
end for
$M = \varnothing$ {We compute the set M by the lattice I ; initially, $M = \varnothing$}
for all $\alpha \in I$ do
$C = \varnothing$ {We compute $C_\alpha = \{x_j \mid p_j \in \alpha\} \cup \{y_j \mid p_j \in \alpha\}$}
for $j = 1, n$ do
if $p_j \in \alpha$ then
$C = C \cup \{x_j\}$
else
$C = C \cup \{y_j\}$
end if
end for
$M = M \bigcup \{C\}$ {Each time we get a minimal vertex cover C_α , we add it to M }
end for

We give an implementation of the previous algorithm in Turbo Pascal Version 7.0.

```
program Lattice_associated_to_a_Cohen_Macaulay_bipartite_graph;
uses Crt;
type
       vector=array[1..100] of integer;
       relation=array[1..100,1..100] of integer;
       m_int=set of 0..10;

var
     i,j,k,l,m,n:integer;
     level:vector;
     R:relation;
     ML:m_int;
     IP:array[1..10,1..1024] of m_int;
begin
ClrScr;
writeln('Introduce the cardinality of the poset');
```

```
write(‘n=’);readln(n);
writeln(n);
 for i:=1 to n do
   for j:=1 to n do
    begin
     if i=j thenR[I,j]:=1
            else R[I,j]:=0;
    end;
writeln(‘introduce the number of the relations of type p[i]<p[j] implies i<j’);
write(‘m=’);readln(m);
writeln(‘introduce the relations of type p[i]<p[j] implies i<j’);
for k:=1 to m do
   begin
    write(‘i=’);readln(i);
    write(‘j=’);readln(j);writeln;
    R[i,j]:=1;
    end;
writeln(‘Write all the relations of the poset);
for i:=1 to n do writeln(‘p[‘,i,’]=p[‘,i,’]’);
for i:=1 to n-1 do
   begin
    for j:=i+1 to n do
     if R[i,j]=1 then writeln(‘p[‘,i,’]=p[‘,j,’]’);
   end;
k:=1;
l:=1;
IP[k,l]=[];
l:=l+1;
IP[k,l]:=[1];
level[k]:=l;
if n>1 then
begin
repeat
k:=k+1;
level[k]:=level[k-1];
for l:=1 to level[k] do IP[k,l]:=IP[k-1,l];
ML:=[];
   for j:=1 to k-1do
     if R[j,k]=1 then ML:=ML+[j];
for l:=1 to level[k-1] do
   if ML<=IP[k-1,l] then
     begin
      IP[k,level[k]+1]:=IP[k-1,l]+[k];
      level[k]:=level[k]+1;
     end;
until k=n;
end;
for t:=1 to level[n] do
   begin
     write(‘Poset ideal ‘,t,’:’);
     if IP[n,t]=[] then writeln(‘empty set’)
                   else
                     begin
                       for i:=1 to n dp
```

```
                    if I in IP[n,t] then write(‘p[‘,i,’] ‘);writeln;
                   end;
   end;
  readln;
  writeln;
  for t:=1 to level[n] do
   begin
    write(‘Minimal vertex cover number ‘,t,’:’);
    if IP[n,t]=[] then
                  for i:=1 to n do write(‘y[‘,i,’] ‘)
                 else
                  begin
                   for i:=1 to n do
                    if i in IP[n,t] then write(‘x[‘,i,’] ‘)
                                  else write(‘y[‘,i,’] ‘);
                  end;
  writeln;
  end;
  readln;
  end.
```

Example 2.4. Let $P_4 = \{p_1, p_2, p_3, p_4\}$ be the poset with $p_1 \le p_2$, $p_1 \le p_4$ and $p_2 \le p_3$. The Hasse diagram of P_4 is represented in the next figure:

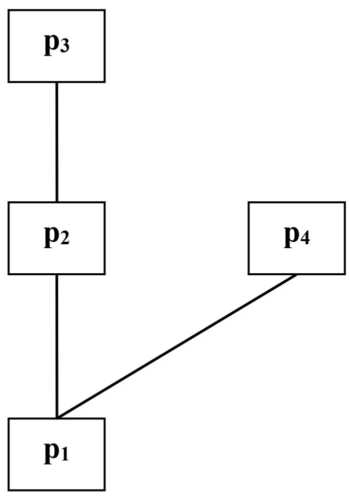

Fig. 1

Let $G = G(P_4)$ be the bipartite graph that comes from the poset P_4. The graph is represented geometrically in the next figure:

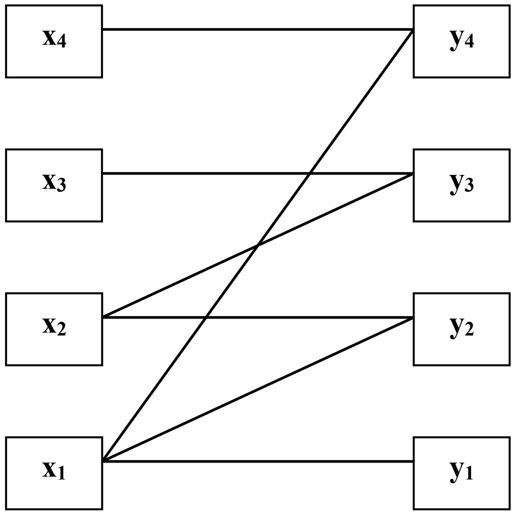

Fig. 2

By using the program based on the Algorithm 2.3. we get the distributive lattice associated to the graph G :

$I(P_4)=\{\varnothing,\{p_1\},\{p_1,p_2\},\{p_1,p_2,p_3\},\{p_1,p_4\},\{p_1,p_2,p_4\},\{p_1,p_2,p_3,p_4\}\}$.

The Hasse diagram of the poset $(I(P_4),\subset)$ is depicted in the next figure:

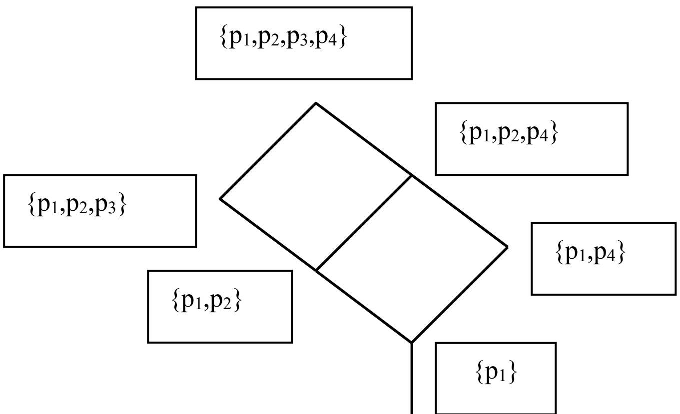

Fig. 3

Finally, we get the set $M(G)$ of all minimal vertex covers of the graph G :

$M(G)=\{\{y_1.y_2,y_3,y_4\},\{x_1,y_2,y_3,y_4\},\{x_1,x_2,y_3,y_4\},\{x_1,x_2,x_3,y_4\},\{x_1,y_2,y_3,x_4\},\{x_1,x_2,y_3,x_4\},$
$\{x_1,x_2,x_3,x_4\}\}$.

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