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GEOMETRIC CHARACTERIZATION AND PARAMETERS SELECTION OF GIELIS SUPERFORMULA-BASED PITCH CURVES FOR UNDERCUT-FREE NONCIRCULAR GEARS DESIGN

Paul-Adrian PASCU, Laurentia ANDREI

"Dunarea de Jos" University of Galati
email: landrei@ugal.ro

ABSTRACT

The paper is concerned with the insertion of the Gielis' supershape in the non-circular gears' theory, by selecting a subset of the modified elliptical shapes to define the pitch curve of the driving gear. A one-factor-at-a-time sensitivity analysis enables a proper selection of the Gielis' supershape defining parameters to achieve convex shapes, with maximum curvature limited by the undercutting phenomenon occurring during gear manufacture. A simulation of the gear's teeth generation, wherein a pinion-type shaper cutter serves as a cutting tool, enables the illustration of the tooth flank profile geometry, providing a graphical validation of the developed analysis. Manipulating the driving pitch curve geometry, through the Gielis' superformula parameters, the transmission ratio variation can be easily achieved.

KEYWORDS: Gielis' superformula, elliptical gears, undercutting

1. INTRODUCTION

The generalized ellipse (Lamé curve), described by a Cartesian equation, is a planar curve whose geometry is controlled by three parameters, i.e., the semiaxes lengths and an exponent-parameter that defines the roundness of the ellipse. Extending this equation, by adding new parameters, and transferring it into polar coordinates, Gielis enabled an extremely wide variety of shapes to be produced, with radial symmetry or natural-looking unevenness [4]. Being simple, compact, and extremely flexible, the Gielis' superformula is largely used in mathematics, computer graphics, engineering, architecture, biology, art design, etc. In mechanical engineering, the Gielis superformula has become a useful instrument for shape optimization [12], parametric CAD-based generative design [5, 6, 10], non-standard cam and gear design [9, 13], aerodynamic and hydrodynamic profile generation [2, 11] etc.

Focusing on the non-circular gears, the Gielis superformula can be nicely used within the geometrical hypothesis of the gears' design, where the driving gear's pitch curve geometry is the initially defined data. Following this hypothesis, the paper is concerned with the analysis of the influence the Gielis' parameters have on the conjugate centrodes geometries and further gear tooth generation. Given that i) the Gielis' superformula allows the definition of a virtually infinite set of curves and ii) the elliptical gears are one of the most widely explored type of non-circular gears, described and generated through different mathematical equations and methods [1, 3, 7, 8], the study is restricted to the family of conventional and modified ellipses chosen for the "supershaped" driving gear pitch curve.

Using Microsoft Excel and AutoCAD/ AutoLISP as computing and graphic tools, the algorithm of the proposed analysis addresses the following aspects: I) A short outline of the non-circular gears pitch curves design, wherein the driving gear pitch curve, which is generated from the elliptical family of the Gielis' supershapes, is considered as the initial data; II) Evaluation of the influence the supershape defining parameters have on the conjugate pitch curves curvature and convexity; III) Limitation of the parameters variation ranges by two imposed constraints, namely the preservation of the curve's convexity and the avoidance of undercutting in the subsequent gear generation; IV) Validation of the numerical results through the gear tooth geometry illustration, over the critical zones.

Finally, the effect of the Gielis' superformula parameters on the non-circular gears transmission ratio variation is analysed.

2. THE GIELIS' SUPERSHAPE AS ELLIPTICAL PITCH CURVE

Initially created to mathematically describe natural and biological shapes, the Gielis' superformula [4] defines both shapes showing radial symmetry and asymmetric shapes alike, through six parameters:

$$r(\varphi) = \left(\left| \frac{1}{a} cos \frac{m\varphi}{4} \right|^{n_2} + \left| \frac{1}{b} sin \frac{m\varphi}{4} \right|^{n_3} \right)^{-\frac{1}{n_1}} \quad (1)$$

where r, φ are the polar coordinates, $\varphi \in [0, 2\pi]$; a, b - length-related parameters, governing the length of the conventional ellipse semi-axes; m – a real positive parameter that controls the curve's rotational symmetry; n_2, n_3 – real positive exponent-parameters that locally influence the shape curvature along the principal X and Y axes, respectively; n_1 – a real positive non-zero exponent parameter that controls the overall shape's smoothness or sharpness.

In case of the following values of the parameters: $m = 4$, $n_1 = n_2 = n_3 = 2$, if $a \neq b$, a standard ellipse (also called a conventional ellipse) is defined by the Gielis' superformula; modifying one parameter, a subset of the elliptical shape family is generated. Consequently, the authors propose further expanding the existing database on modified elliptical gears geometry by using the Gielis' supershape as a potential pitch curve. In the case of conjugate gears, this will be the driving gear pitch curve. The influence of the parameter variations on the geometries of both the driving and driven pitch curves is the main subject to be considered. Keeping the m parameter at fixed value, $m = 4$, and introducing the term of "normalized centrode" for the Gielis' supershape with the semi-axes lengths being varied in the vicinity of unit values, the polar definition of the normalized centrode becomes:

$$r_1^*(\varphi_1) = \left(\left| \frac{1}{a} cos\varphi_1 \right|^{n_2} + \left| \frac{1}{b} sin\varphi_1 \right|^{n_3} \right)^{-\frac{1}{n_1}} \quad (2)$$

where $\varphi_1 \in [0, 2\pi]$.

To insert the normalized centrode into the non-circular gears' theory, this needs to be scaled so that its length matches the dimensional requirement of a gear pitch curve defined by its specific parameters, namely the module m and the number of teeth z_1. As a consequence, the definition of the driving gear pitch curve will be written as:

$$r_1(\varphi_1) = \lambda_{sc} \cdot r_1^*(\varphi_1), \varphi_1 \in [0,2\pi] \quad (3)$$

where λ_{sc} is the scale factor,

$$\lambda_{sc} = \frac{\pi m z_1}{L_1^*}, \quad (4)$$

L_1^* being the length of the normalized driving centrode,

$$L_1^* = \int_0^{2\pi} \sqrt{[r_1^*(\varphi_1)]^2 + \left[\frac{dr_1^*(\varphi_1)}{d\varphi_1}\right]^2} \quad (5)$$

Following the well-known procedure in the non-circular gear generation theory [7], the driven pitch curve is defined by the polar coordinates:

$$r_2(\varphi_2(\varphi_1)) = A - r_1(\varphi_1) \quad (6)$$

$$\varphi_2(\varphi_1) = \int_0^{\varphi_1} \frac{r_1(\varphi)}{A - r_1(\varphi)} d\varphi \quad (7)$$

where A is the constant gears center distance.

Assuming that both conjugate gears have closed pitch curves and perform one revolution during the motion period, the center distance is obtained through an iterative procedure, from the equation:

$$L_1^* = \int_0^{2\pi} \sqrt{[r_1^*(\varphi_1)]^2 + \left[\frac{dr_1^*(\varphi_1)}{d\varphi_1}\right]^2} \quad (8)$$

3. GEOMETRIC ANALYSIS OF THE GIELIS' SUPERFORMULA-BASED PITCH CURVES

The most informative descriptor for the non-circular pitch curves geometry is the radius of curvature, providing critical insights into the feasibility, mechanical performance, and manufacturability of the gear. Explicitly, to sustain a further realistic gear profiles generation, it is necessary to investigate the radius of curvature through the following aspects: i) its sign and amplitude, enabling to depict the occurrence of concave regions or lobes, and ii) its correlation with the risk of undercutting constraint.

To avoid undercutting, when non-circular gears are manufactured using classical rolling generation, the minimum local radius of curvature should be larger than a limit radius determined only by the rack parameters [n]:

$$\rho_{min} \geq \frac{m \cdot h_a^*}{(\sin \alpha_c)^2} \quad (9)$$

where m is the gears modulus, h_a^* - the coefficient of the tooth addendum (for a standard gear tooth, $h_a^* = 1$), α_c – the pressure angle.

The pitch curve's radius of curvature could be substituted by the pitch curve's curvature. In this context, also considering the analytical expression for the curve's curvature, the above equation becomes:

$$k_{max}(\varphi) = \frac{[r(\varphi)^2 + 2r'(\varphi)^2 - r(\varphi) \cdot r''(\varphi)]}{[r(\varphi)^2 + r'(\varphi)^2]^{\frac{3}{2}}} \leq \frac{(sin\,\alpha_c)^2}{m \cdot h_a^*} \quad (10)$$

where r' and r'' are the first and second derivative of the curve' s definition function r, respectively.

The occurrence of undercutting may be graphically assessed by simulating the gear tooth generation process and by identifying the tooth flanks' profile in critical regions where the maximum curvature fails to satisfy the condition from relation (10).

4. NUMERICAL CASE STUDY

The study is concerned with the geometry analysis in case of conjugated pitch curves specific to a non-circular gear train with the gear ratio N_1:N_2 = 1:1. The driving pitch curve derives from an elliptical subset of the Gielis' supershape (Eqs. 3-6) and is assigned with a variable geometry, through the defining parameters a, b, n_1, n_2, n_3; accordingly, the driven pitch curve' s geometry varies, defined by Eqs. 7-9.

The specific gear parameters are chosen as: module m = 3 mm, number of teeth z_1 = z_2 = 36, pressure angle α_c = 20°, and the coefficient of the tooth addendum h_a = 1. Additional assumptions are considered in the proposed analysis, namely: i) the exponent parameters are considered in the vicinity of the value 2, specific to the standard ellipse, in order to avoid sharp corners, and ii) the semi-axis length b is considered equal to 1.

A one-factor-at-a-time sensitivity analysis is applied and the maximum curvature variations, for both driving and driven pitch curves, related to the limit that would imply the occurrence of undercutting in the gear teeth manufacturing, is the main geometrical concern. In the final stage, the investigations are validated by tooth-profile plots, obtained via the simulation of the gear manufacture, developed in AutoCAD/AutoLISP, wherein the rolling method is used and a pinion-type shaper cutter, with modulus m = 3 mm and number of teeth z = 20.

As a first step, the driving centrode semi-axis length, a parameter, is varied, while the exponent parameters are kept at constant equal values: $n_1 = n_2 = n_3 = 2$ (defining the conventional ellipse). As seen in Figure 1, as expected, the higher the a length-parameter value, the higher the maximum curvature (the lower the radius of curvature) on both pitch curves.

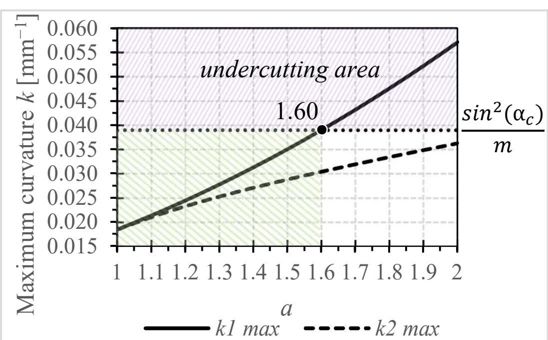
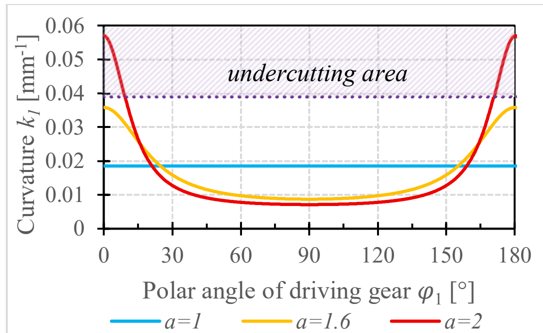
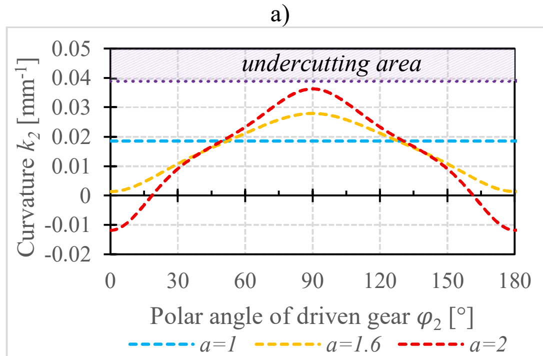


Fig. 1. *Maximum curvatures of the driving (k_{1max}) and driven (k_{2max}) pitch curves as functions of the semi-axis length, a, in case of $n_1 = n_2 = n_3 = 2$*

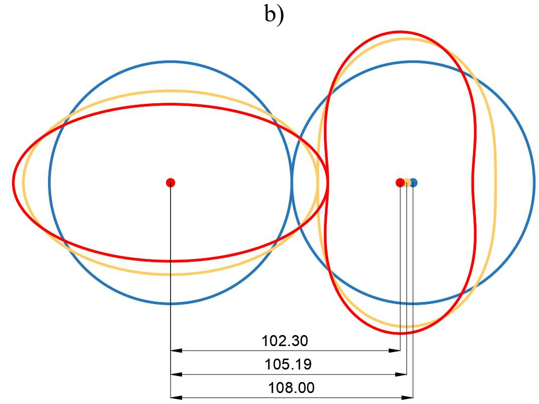
On the driving element, the maximum curvature exhibits a faster increase, compared to the driven pitch curve's curvature, so it is expected that undercutting will first occur on the driving gear, while $a \geq 1{,}6$ mm. In Figure 2, both conjugate pitch curves' curvature variations (fig. 2a, b) and the resulting curves are illustrated, with the center distance specification (fig. 2c), considering three cases: I. circular pitch curves (a = 1 mm); II. elliptical shapes (a = 1,6 mm), with the driving pitch curve having a limiting geometry, due to the local maximum of the curvature that reaches the undercutting limit, at 0° and 180° polar angles; III. elliptical shapes (a = 2 mm), with undesirable geometries since the driving pitch curve's curvature exceeds the limit of undercutting and negative curvatures/ concave zones appear on the driven centrode.



a)



b)



c)

Fig. 3. *Geometrical aspects of conjugate pitch curves in case of $n_1 = n_2 = n_3 = 2$ and variable semi-axes length, a: variations of curvature (a, b), shapes (c)*

On the driving pitch curve, it can be noticed that the higher the semi-axis length value, the wider the range of curvature variation and the higher its rate of change, within a relatively small angular interval in the vicinity of points with the maximum curvature recorded. The driven pitch curve curvature exhibits a more gradual variation occuring over a wider angular interval. Although the maximum recorded curvature is underneath the undercutting limit, for a = 2 mm, the negative curvature values could create problems for further tooth generation. The profile of the driving gear's tooth, from the region of maximum curvature, is illustrated in Fig. 3, to graphically validate the analytical results, i.e, correct tooth profiles (case II) and tooth exhibiting profile undercutting (case III).

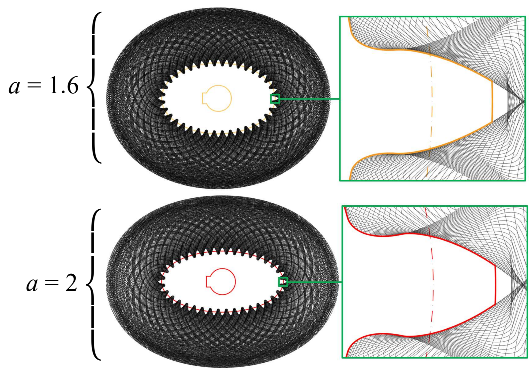


Fig. 3. *Tooth profile comparison when "a" varies*

To analyse the influence of the global exponent-parameter, n_1, on the conjugate pitch curves geometries, the driving normalized centrode semi-axis length is considered as a = 1,4 mm (value that would enable proper teeth generation as shown in previous analysis), while the local exponent-parameters are kept at constant equal values: $n_2 = n_3 = 2$. As seen in Figure 4, increasing the n_1 parameter above the minimum value of n_1 = 1,2 (the lower bound for undercutting onset in the driving gear) reduces the maximum curvature of both pitch curves.

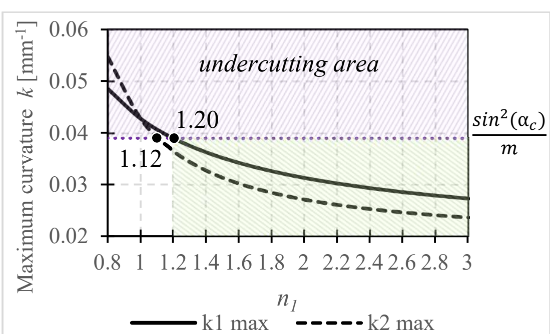


Fig. 4. *Maximum curvatures of the driving (k_{1max}) and driven (k_{2max}) pitch curves as functions of the n_1 parameter, in case of a = 1,4* mm, *$n_2 = n_3 = 2$*

In Figure 5, three cases are evaluated: IV) both conjugate pitch curves have unproper shapes for further teeth manufacture (n_1 = 1); V) the driving pitch curve has the „limit" elliptical shape (n_1 = 1,2), the local maximum of the curvature reaching the undercutting avoidance value; VI) both conjugate pitch curves are suitable for teeth manufacture (n_1 = 2). As this global exponent parameter controls the

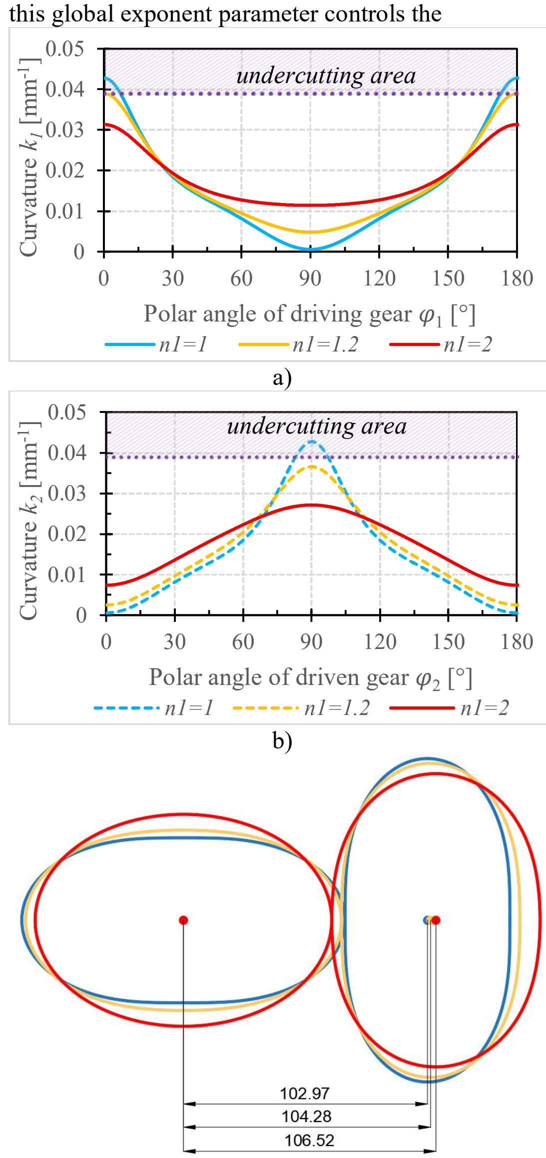


Fig. 5. *Illustrating geometrical aspects of conjugate pitch curves in case of a = 1,4 mm, $n_2 = n_3 = 2$ and variable n_1: variations of curvature (a, b), shapes (c)*

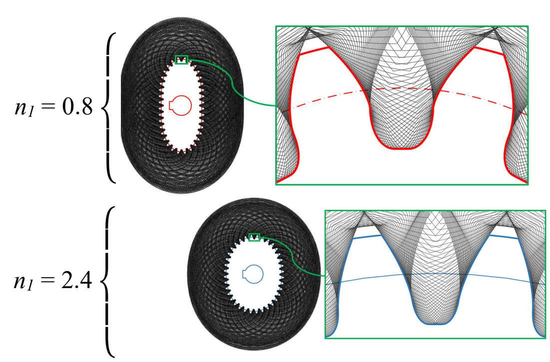


Fig. 6. *Tooth profile comparison when "n_1" varies*

supershape smoothing, higher exponent values are confirmed to decrease the maximum curvature amplitude and the curvature variation range and rate, as seen in Figures 5a, b. The smoothing effect of the exponent on the pitch curve geometries is visible in Figure 5c. In Fig. 6, there are illustrations describing the driven gear teeth in the vicinity of the polar angle $\varphi_2 = 90°$, where a maximum curvature is recorded. For $n_1 = 0{,}8$, the driven gear exhibits curvature with a value below the limit of undercutting (fig. 4), and the gear teeth profiles' geometry is modified, as expected; for $n_1 = 2{,}4$ a proper tooth geometry is generated.

As regards the local exponent-parameters n_2 and n_3 from the Gielis' superformula, since n_2 *is* linked to the term that includes the parameter $a \geq 1$ that influences the cosine function, it is obvious that the variation of the n_2 parameter will significantly affect the gears' geometry and will be subjected to analyses. Plotting the maximum curvature of both conjugate pitch curves, when n_2 is varied, a narrow range for the parameter's allowable variation, especially induced by the driven pitch curve's geometry, is noticed. With n_2 values smaller than the "standard" value ($n_2 = 2$), the driven pitch curve would exhibit negative curvature and concave zones, respectively, with potential unfavourable effect on the gear tooth geometry; a parameter $n_2 \geq 2{,}39$ induces a driven pitch curve's geometry which results in tooth undercutting.

Increasing the n_2 value, the driving pitch curve curvature would exhibit a sharp variation in value (fig. 8a); at 90° and 270° polar angles, the maximum curvature converges to zero and almost linear zones would complete the pitch curve geometry (fig. 8c). For the driven gear, the increase of n_2 value leads to zones almost linear on its pitch curve, in the vicinity of 0° and 180° polar angles, and narrow zones with high curvatures, at 90° and 270°, where the tendency toward undercutting becomes more pronounced (fig. 8b). In figure 9, teeth chosen on the driven gears, specific to critical zones, are illustrated. Following the curvature variation from Fig. 7, the n_2 parameter's values were selected to generate both correct teeth and undercutting, for graphical confirmation of the numerical data.

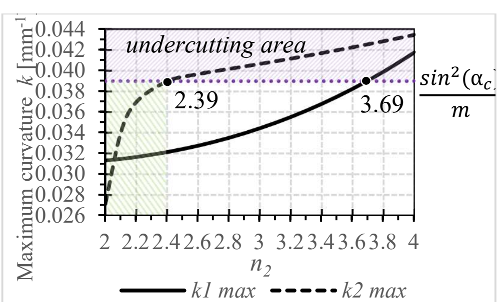


Fig. 7. *Maximum curvatures of the driving (k_{1max}) and driven (k_{2max}) pitch curves as functions of the n_2 parameter, in case of $a = 1{,}4$* mm, *$n_1 = n_3 = 2$*

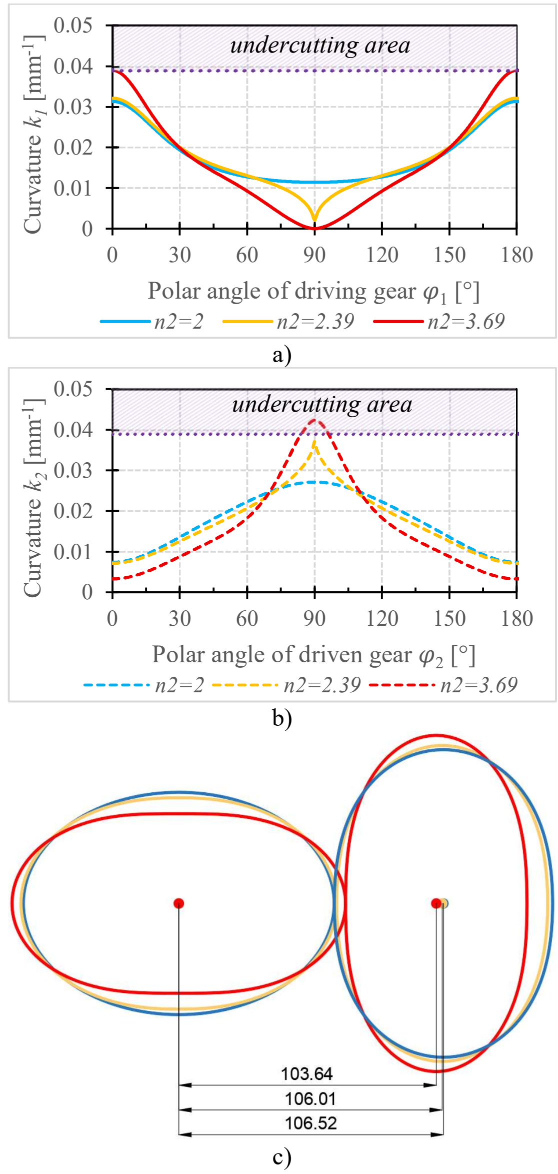


Fig. 8. *Illustrating geometrical aspects of conjugate pitch curves in case of $a = 1{,}4$ mm, $n_1 = n_3 = 2$ and variable n_2: variations of curvature (a, b), shapes (c)*

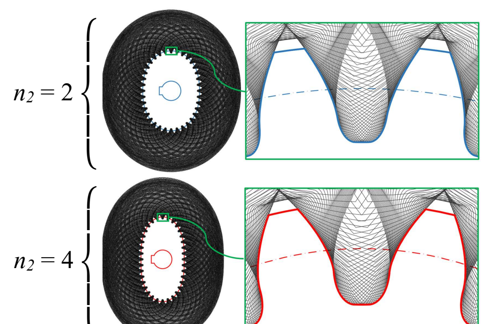


Fig. 9. *Tooth profile comparison when "n_2" varies*

5. GEARS TRANSMISSION RATIO

In case of non-circular gears, the instantaneous transmission ratio could be defined in terms of the instantaneous pitch radii, at the contact point, as:

$$i_{12} = \frac{r_2(\varphi_2)}{r_1(\varphi_1)} = \frac{A - r_1(\varphi_1)}{r_1(\varphi_1)} \quad (11)$$

Modifying the pitch curves geometries, through Gielis' superformula parameters, the influence on the transmission ratio variation is obviously an important issue, as the kinematic behavior is the priority of the non-circular gear design. For the chosen parameters previously presented, the variation of the gear's transmission ratio is illustrated in Fig. 10. An increase in parameter a clearly results in an increased amplitude of the transmission ratio. The parameter n_1 has an even greater impact on the transmission ratio, but in the opposite direction compared to a: as its value increases, the amplitude of the transmission ratio decreases. The influence of n_2 is limited in our case by undercutting constraints, resulting in a smaller range of variation.

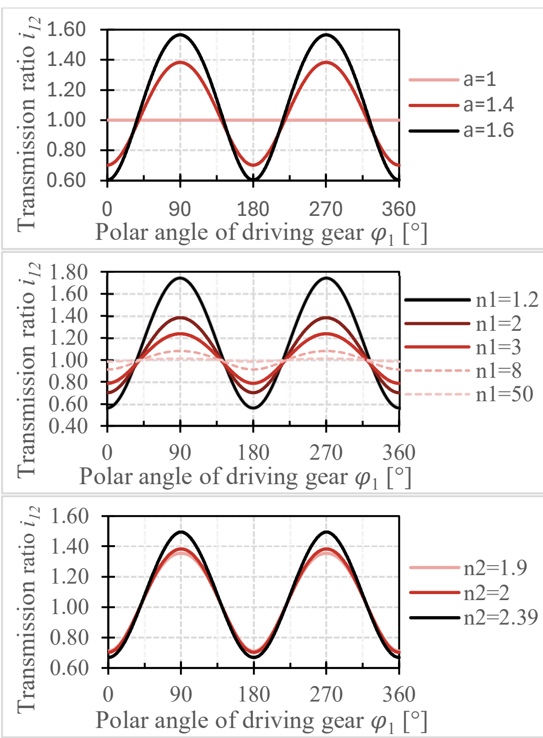


Fig. 10. *Influence of a, n_1, n_2 parameters on the transmission ratio variation*

6. CONCLUSIONS

In this paper, the Gielis' superformula is chosen to describe conventional and modified ellipses that will be assigned, as pitch curves, to the driving gear of a non-circular gear train. Depending on several parameters, the driving pitch curve geometry is investigated through parameter variation in order to preserve convexity and avoiding undercutting during subsequent tooth generation. Following the presentation of the general theoretical background, a numerical study was performed, based on the one-factor-at-a-time analysis, revealing the following:

- An increase of more than 60% in the major semi-axis length, a, leads to undercutting of the teeth positioned on the axis direction. As the major axis is elongated, the transmission ratio variation increases, as expected;
- An increase in global exponent-parameter n_1 results in the rounding of the pitch curves, making them more suitable for tooth profile generation, but it leads to a decrease in the ratio variation amplitude; to increase this amplitude, the parameter could be reduced till the undercutting effect occurs;
- The range of the local exponent-parameter n_2 variation was shown to be extremely limited by the appearance of both sharp corners and undercutting; as a result, a limited influence on the gear transmission ratio variation was recorded.

It is obvious that an arbitrary choice of parameters could lead to an unexpected geometry of the non-circular gear and its generation and analysis would become particularly challenging.

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